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Abstracts of short presentations

Strong R-completeness for the Product-Free Lambek Calculus with Intersection and Unit

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Abstract

The Lambek calculus allowing empty antecedents, extended with intersection, is weakly, but not strongly complete w.r.t. relational models. We show that, in contrast, its product-free fragment enjoys strong completeness. Thus, in the case with empty antecedents, the Lambek calculus with only one conjunction (either product or meet) enjoys strong completeness, while adding both conjunctions ruins it.

Keywords: Lambek calculus, R-models, strong completeness.

1 Introduction

The Lambek calculus [12] is a non-classical logical system which was initially introduced for applications in mathematical linguistics. Later on, the place of the Lambek calculus in the family of logical systems was understood more deeply. On one hand, the Lambek calculus is a fragment of intuitionistic non-commututative version of Girard's [7] linear logic [1]. A modern presentation of the Lambek calculus, its relations with linear logic, and its usage in linguistics is presented in the book by Moot and Retoré [17]. The connection with linear logic motivates extending the Lambek calculus with linear logic connectives, in particular, additive conjunction and disjunction (meet and join) [13,3,9]. This extension is called the multiplicative-additive Lambek calculus (MALC).

On the other hand, MALC is actually one of the most basic substructural logics, and it has a natural class of algebraic models—residuated lattices (see [6]). Heyting algebras form a subclass of residuated lattices (Heyting algebras are residuated lattices where product and meet coincide). Therefore, MALC is a weaker logic than intuitionistic propositional logic. Namely, structural rules of weakening, contraction, and even permutation are removed (this motivates the term "substructural"). By a standard argument one shows algebraic soundness and completeness for MALC.

One of the natural specific classes of residuated lattices is the class of algebras of binary relations. (Another natural class is the class of algebras of

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formal languages, which reflects the linguistical meaning of the Lambek calculus; however, we focus on algebras of binary relations.) Being considered as models for MALC and its fragments [2], models on algebras of binary relations (R-models for short) raise interesting (in)completeness questions, in comparison with general algebraic models, for which completeness is easy.

Namely, R-models enjoy distributivity for additive conjunction and disjunction, which is not derivable in MALC. More accurate analysis shows that the actual problem is with join: there exists a corollary of the distributivity law which does not use meet and which is not derivable in MALC [10]. This means that the Lambek calculus extended with join only is already incomplete w.r.t. R-models. Moreover, there are negative results concerning strong completeness (i.e., completeness for derivability from sets of hypotheses) for axiomatic extensions of MALC w.r.t. R-models [2]. In particular, adding the distributivity law as an extra axiom would not make MALC strongly complete w.r.t. R-models.

For this reason, further we extend the Lambek calculus with meet only. Since in R-models meet is interpreted as set-theoretic intersection, we call this operation "intersection." Another motivation for intersection is the analogy with type systems for the λ-calculus. Here we see a reminiscence with intersection types by Coppo and Dezani [5] and Sallé [19] (see also [8]). The Lambek calculus may also be viewed as a type system, though a substructural one.

The result presented is an increment to the line of research of Andréka and Mikulás [2,15,16] and the author [11]. For the system with the unit and intersection there is only weak, but not strong completeness w.r.t. R-models. We show that strong completeness is restored if one removes product (multiplicative conjunction). The same happens without additive conjunction (intersection) [2].

2 Calculi and Models

We are going to define extensions of the Lambek calculus and its variants with intersection $\wedge$ (as a binary operation) and the unit constant $\mathbf{1}$. In fact, the unit constant adds some mess into relational semantics, but we shall need it in the proofs. We formulae the systems in questions as sequent calculi.

Formulae (denoted by capital Latin letters) are constructed from a countable set of variables $\mathrm{Var} = \{p, q, r, \ldots\}$ and constant $\mathbf{1}$ using four binary connectives: $\cdot$ (product), $\backslash$ and $/$ (left and right divisions), and $\wedge$ (intersection). The set of formulae is denoted by Fm. Capital Greek letters stand for sequences of formulae, Λ being the empty one. Sequents are expressions of the form $\Pi \to C$, where Π and C are called the antecedent and the succedent respectively.

All sequents of the form $A \to A$ are axioms. Inference rules of the original Lambek calculus $\mathbf{L}$ [12] are as follows. [2]

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi, A \backslash B, \Delta \to C} \backslash L \qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B} \backslash R, \text{ where } \Pi \neq \Lambda$$

[2] Here all formulae should be in the language of $\cdot$, $\backslash$, $/$, i.e., without $\mathbf{1}$ and $\wedge$. When talking about extensions of $\mathbf{L}$, we shall extend the formula language accordingly, which will retrospectively apply to axioms and "old" rules as well.

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, B \,/\, A, \Pi, \Delta \to C} \;/\, L \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A} \;/\, R, \text{ where } \Pi \neq \Lambda$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C} \;\cdot L \qquad \frac{\Pi \to A \quad \Delta \to B}{\Pi, \Delta \to A \cdot B} \;\cdot R \qquad \frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C} \; Cut$$

This calculus has a distinctive feature called Lambek's non-emptiness restriction: the $\Pi \neq \Lambda$ constraints on $\setminus R$ and $/\, R$ rules make antecedents of all derivable sequents non-empty. Lifting this restriction yields the Lambek calculus with empty antecedents [13], which we denote by $\mathbf{L}^{\Lambda}$.

This calculus can be further extended with the unit constant $\mathbf{1}$, by adding axiom $\Lambda \to \mathbf{1}$ (which serves as the $\mathbf{1}R$ rule) and the following $\mathbf{1}L$ rule:

$$\frac{\Gamma, \Delta \to C}{\Gamma, \mathbf{1}, \Delta \to C} \; \mathbf{1}L$$

The resulting system is the Lambek calculus with the unit, denoted by $\mathbf{L}^{\mathbf{1}}$. (The unit constant is incompatible with Lambek's non-emptiness restriction, since the axiom for it has an empty antecedent. Thus, adding $\mathbf{1}$ to $\mathbf{L}$ instead of $\mathbf{L}^{\Lambda}$ is meaningless.)

Finally, each of the systems $\mathbf{L}$, $\mathbf{L}^{\Lambda}$, $\mathbf{L}^{\mathbf{1}}$ can be extended by intersection governed by the following rules:

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} \quad \frac{\Gamma, B, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} \;\wedge L \qquad \frac{\Pi \to A \quad \Pi \to B}{\Pi \to A \wedge B} \;\wedge R$$

These extensions will be denoted by $\mathbf{L}\wedge$, $\mathbf{L}^{\Lambda}\wedge$, $\mathbf{L}^{\mathbf{1}}\wedge$ respectively.

An R-model (more precisely, a relativised R-model) is a triple $\mathcal{M} = (W, U, v)$, where W is an arbitrary set, $U \subseteq W \times W$ is the so-called universal relation, which is required to be transitive, and $v \colon \mathrm{Var} \to \mathcal{P}(U)$ is a valuation function (each variable is interpreted by a subrelation of U). This function is propagated to formulae without the unit, giving the interpreting function $\bar{v}$:

$$\bar{v}(p) = v(p),\; p \in \mathrm{Var} \qquad \bar{v}(A \wedge B) = \bar{v}(A) \cap \bar{v}(B)$$
$$\bar{v}(A \setminus B) = \{(y, z) \in U \mid (\forall x \in W)\, ((x, y) \in \bar{v}(A) \Rightarrow (x, z) \in \bar{v}(B))\}$$
$$\bar{v}(B \,/\, A) = \{(x, y) \in U \mid (\forall z \in W)\, ((y, z) \in \bar{v}(A) \Rightarrow (x, z) \in \bar{v}(B))\}$$
$$\bar{v}(A \cdot B) = \bar{v}(A) \circ \bar{v}(B) = \{(x, z) \mid (\exists y \in W)\, ((x, y) \in \bar{v}(A) \text{ and } (y, z) \in \bar{v}(B))\}$$

Notice that the interpretation for division operations essentially depends on U. A square (unrelativised) R-model is an R-model with $U = W \times W$.

A sequent of the form $A_1, \ldots, A_n \to B$, where $n > 0$, is true in $\mathcal{M}$, if $\bar{v}(A_1) \circ \ldots \circ \bar{v}(A_n) \subseteq \bar{v}(B)$. For $n = 0$, the sequent $\Lambda \to B$ is true in $\mathcal{M}$, if $\delta = \{(x, x) \mid x \in W\} \subseteq \bar{v}(B)$.

Let us recall some basic definitions. A logic $\mathcal{L}$ induces the derivability relation, denoted by $\mathcal{H} \vdash_{\mathcal{L}} \Pi \to A$, where $\Pi \to A$ is a sequent and $\mathcal{H}$ is a set of sequents. On the other hand, a class of models $\mathcal{C}$ induces the notion of semantic entailment: $\mathcal{H} \vDash_{\mathcal{C}} \Pi \to A$ means that $\Pi \to A$ is true in any model $\mathcal{M} \in \mathcal{C}$, in which all sequents from $\mathcal{H}$ are true. Next, logic $\mathcal{L}$ is strongly sound w.r.t. class of models $\mathcal{C}$ if $\vdash_{\mathcal{L}}$ entails $\vDash_{\mathcal{L}}$, and it is strongly complete, if the opposite implication holds. Weak soundness and completeness is defined similarly, but with $\mathcal{H} = \varnothing$; the calculi in question do not enjoy deduction theorem, whence weak completeness is essentially weaker than strong one (even for finite $\mathcal{H}$'s).

The correspondence between logics without the unit and their classes of models is as follows. For $\mathcal{L} \in \{\mathbf{L}, \mathbf{L}\wedge\}$ we take as $\mathcal{C}$ the class of relativised R-models and for $\mathcal{L} \in \{\mathbf{L}^{\Lambda}, \mathbf{L}^{\Lambda}\wedge\}$ the class of square ones. Each logic is strongly sound w.r.t. the corresponding class of models. For completeness, the situation is more involved. For $\mathbf{L}$ and $\mathbf{L}\wedge$, strong completeness w.r.t. relativised R-models was proved by Andréka and Mikulás [2]. They also proved strong completeness for $\mathbf{L}^{\Lambda}$ w.r.t. square R-models; however, this proof does not extend to $\mathbf{L}^{\Lambda}\wedge$. For $\mathbf{L}^{\Lambda}\wedge$, Mikulás [15,16] proved weak completeness and suggested a possible counterexample to the strong one; the latter was proved to be indeed a counterexample in [11].

Extending the models for $\mathbf{L}^{\Lambda}$ and $\mathbf{L}^{\Lambda}\wedge$ to $\mathbf{L}^{\mathbf{1}}$ and $\mathbf{L}^{\mathbf{1}}\wedge$ requires non-standard interpretations of the unit: if one defines $\bar{v}(\mathbf{1}) = \delta$, this will result in incompleteness (even in the weak sense) [4]. In a non-standard interpretation, introduced in [11], we have $\bar{v}(\mathbf{1}) = \mathbf{1}_{\mathcal{M}}$, where $\mathbf{1}_{\mathcal{M}}$ is a binary relation on W satisfying $\mathbf{1}_{\mathcal{M}} \circ \bar{v}(A) = \bar{v}(A) = \bar{v}(A) \circ \mathbf{1}_{\mathcal{M}}$ for each $A \in \mathrm{Fm}$. For such extensions of square R-models, weak completeness of $\mathbf{L}^{\mathbf{1}}\wedge$ (and thus $\mathbf{L}^{\Lambda}\wedge$) was proved in [11].

In the family of models for the Lambek calculus and its extensions, R-models, both square and relativised ones, form specific subclasses of general algebraic models (see [6]): namely, residuated monoids or semigroups, possibly with meet-semilattice structure added. Since the class of R-models is narrower than the class of general algebraic models, completeness results for R-models are stronger and therefore more interesting. Another specific class of models for the Lambek calculus is the class of language models, or L-models [18]. In other words, these are models on powersets of free monoids or semigroups. L-models reflect the linguistic meaning of Lambek formulae; however, they are out of the scope of this paper.

3 Product-Free Fragments

In this short paper, we focus on product-free fragments of $\mathbf{L}^{\Lambda}\wedge$ and $\mathbf{L}^{\mathbf{1}}\wedge$, that is, subsystems without $\cdot L$ and $\cdot R$. Considering such fragments is motivated by the fact that the counterexample to strong R-completeness essentially uses the product. And indeed, without it we gain strong completeness:

Theorem *The product-free fragment of* $\mathbf{L}^{\mathbf{1}}\wedge$ *is strongly complete w.r.t. square R-models with non-standard interpretations of* $\mathbf{1}$*. Therefore, the product-free fragment of* $\mathbf{L}^{\Lambda}\wedge$ *is strongly complete w.r.t. square R-models.*

Let us now briefly sketch some ideas of the proof of this theorem. This proof is a modification of the proof of weak R-completeness of $\mathbf{L}^{\mathbf{1}}\wedge$ from [11] (which is, in its turn, a modification of the argument of Andréka and Mikulás [2]). In what follows, $\vdash$ means $\vdash_{\mathbf{L}^{\mathbf{1}}\wedge}$.

The core construction for proving R-completeness is a labelled infinite graph $G = (W, E, \ell)$, where W is a set, $E \subseteq W \times W$, and $\ell \colon E \to \mathrm{Fm}$. This construction from [11], inspired by [2], has the following properties:

(i) E is transitive, and if $(x,y),(y,z) \in E$, then $\vdash \ell(x,z) \to \ell(x,y) \cdot \ell(y,z)$;

(ii) E is reflexive, and $\ell(x,x) = \mathbf{1}$ for each x;

(iii) E is antisymmetric;

(iv) for each $y \in W$ and $A \in \mathrm{Fm}$ there exists such $x \in W$ that if $(y,z) \in E$ then $(x,z) \in E$ and $\ell(x,z) = A \cdot \ell(y,z)$;

(v) for each $y \in W$ and $A \in \mathrm{Fm}$ there exists such $z \in W$ that if $(x,y) \in E$ then $(x,z) \in E$ and $\ell(x,z) = \ell(x,y) \cdot A$;

(vi) if $(x,z) \in E$ and $\vdash \ell(x,z) \to B \cdot C$ then there exists such $y \in W$ that $(x,y),(y,z) \in E$, $\vdash \ell(x,y) \to B$, and $\vdash \ell(y,z) \to C$.

Following [2], we try to make this construction work for strong completeness, by replacing "$\vdash$" by "$\mathcal{H} \vdash$ " in the construction. However, we fail to establish property (vi) in the $x = z$ case. [3]

In order to overcome this issue, we build a graph G which enjoys properties (i)–(v) with $\mathcal{H}$ added, omitting the problematic one. Next, we construct the $\mathcal{H}$-canonical model $\mathcal{M}_{\mathcal{H}}$ with the W from G and $v(p) = \{(x,y) \in E \mid \mathcal{H} \vdash \ell(x,y) \to p\}$ (for each $p \in \mathrm{Var}$); $\mathbf{1}_{\mathcal{M}_{\mathcal{H}}} = \{(x,y) \in E \mid \mathcal{H} \vdash \ell(x,y) \to \mathbf{1}\}$. For product-free formulae A we get $\bar{v}(A) = \{(x,y) \in E \mid \mathcal{H} \vdash \ell(x,y) \to A\}$. The argument here copies the one from [11]: the only case which needed property (vi) of graph G was the case of $A = B \cdot C$.

Now let $\mathcal{H} \nvdash \Lambda \to A$, where $\mathcal{H}$ and A is product-free; sequents with non-empty antecedents are reduced to this case by $\backslash R$. Then for any $x \in W$ we have $\ell(x,x) = \mathbf{1}$ and $(x,x) \notin \bar{v}(A)$ (since $\mathcal{H} \nvdash \mathbf{1} \to A$). Therefore, $\Lambda \to A$ is not true in $\mathcal{M}_{\mathcal{H}}$. On the other hand, each sequent from $\mathcal{H}$ is. Thus, $\mathcal{M}_{\mathcal{H}}$ is the necessary countermodel.

The final issue to be resolved is the issue of conservativity. In the argument presented above, the derivation $\mathcal{H} \vdash \Lambda \to A$ could have used $\cdot$ inside, though $\mathcal{H}$ and A were product-free (due to *Cut*). However, any such derivation can be made product-free by using the exponential modality from linear logic. This modality, along with division operations, allows a form of deduction theorem (see [14] for more details) which allows internalising finite $\mathcal{H}$'s, and, on the other hand, the resulting system enjoys cut elimination, which yields the subformula property and thus conservativity. [4]

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[3] Namely, in this case by antisymmetry we need $y = x = z$, and this requires deducing $\mathcal{H} \vdash \mathbf{1} \to B$ (since $\ell(x,y) = \ell(y,z) = \mathbf{1}$) and $\mathcal{H} \vdash \mathbf{1} \to C$ from $\mathcal{H} \vdash \mathbf{1} \to B \cdot C$. This indeed holds for $\mathcal{H} = \varnothing$, but fails for non-empty $\mathcal{H}$'s. In [2] for $\mathbf{L}\wedge$, E was irreflexive, so this problem did not arise.

[4] A posteriori, conservativity also follows from strong R-completeness.

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On the Priorean Temporal Logics of Trees

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Abstract

We report some results on axiomatizing the modal and temporal logics of a range of natural classes of trees. We have obtained (weak) completeness results for the modal logics and Priorean temporal logics of the classes of trees obtained by imposing most of the consistent combinations of the properties of discreteness, local fineness, left and right unboundedness, (converse) well-foundedness, density, completeness, reflexivity and irreflexivity. Here we summarize these results and sketch a proof of the completeness result for one of the most difficult cases, the logic of the class of trees with all branches isomorphic to the real line.

Keywords: Axiomatisation, trees, real trees, Priorean temporal logic.

1 Introduction

Tree-like structures are ubiquitous in mathematics and computer science, where they naturally arise in various contexts. Trees, representing branching time structures, are especially useful as models of time where the past is assumed to be fixed but the future is undetermined. Depending on the intended applications, or on specific philosophical assumptions about the nature of time, one may want to impose different additional conditions on tree-like models. For example, in an application where time is measured to a limited accuracy, a discrete ordering may be appropriate, while for others it might be more suitable to consider dense orderings.

Here we report some results on a project aiming at axiomatizing the modal and temporal logics of a range of natural classes of trees. We have obtained (weak) completeness results for the modal logics and Priorean temporal logics of the classes of trees obtained by imposing most of the consistent combinations of the properties of discreteness, local fineness, left and right unboundedness, (converse) well-foundedness, density, completeness, reflexivity and irreflexivity. We summarize these results and sketch a proof of the completeness result for the logic of the class of trees with all branches isomorphic to the real line, which is one of the most difficult cases.

$\mathbf{K_G}$	$\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$	$\mathbf{S}$	$\neg p \wedge \mathbf{H}\neg p \wedge \mathbf{F}p \to \mathbf{F}(p \wedge \mathbf{H}\neg p)$
$\mathbf{K_H}$:	$\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$	$\mathbf{U_l}$	$\mathbf{P}\top$
$\mathbf{Dual_F}$	$\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$	$\mathbf{U_r}$	$\mathbf{F}\top$
$\mathbf{Dual_P}$	$\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$	$\mathbf{L_l}$	$\mathbf{H}(\mathbf{H}p \to p) \to \mathbf{H}p$
Conv	$p \to \mathbf{GP}p$ and $p \to \mathbf{HF}p$	$\mathbf{L_r}$	$\mathbf{G}(\mathbf{G}p \to p) \to \mathbf{G}p$
4	$\mathbf{FF}p \to \mathbf{F}p$	**T**	$p \to \mathbf{F}p$
$\mathbf{.3_l}$	$(\mathbf{P}p \wedge \mathbf{P}q) \to$ $\mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$	**Grz**	$\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to p$
D	$\mathbf{F}p \to \mathbf{FF}p$	$\mathbf{Grz_l}$	$\mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to p$

Table 1
Axioms for Priorean temporal logics of trees

2 Preliminaries

A (strict) partial order $(W, \prec)$ is called **left-linear**, if it satisfies $\forall x, y, z(x \prec y \wedge z \prec y \to (x = z \vee x \prec z \vee z \prec x)$; **connected**, if $\forall x, y(x \prec y \vee y \prec x \vee \exists z(z \prec x \wedge z \prec y))$; a **(strict) tree**, if $\prec$ is both left-linear and connected; **forward/backward discrete**, if every instant with a successor/predecessor has an immediate successor/predecessor; **discrete**, if it is both forward and backward discrete; **locally finite**, if there is a finite number of distinct instants between any two comparable instants; **continuous**, if every non-empty subset with an upper bound has a least upper bound. A **branch** of a tree is a maximal (with respect to set inclusion) subset of T that is linearly ordered by $\prec$.

By endowing a tree with a valuation for proposition letters by sets of instants we obtain a Kripke model on which we interpret the basic modal language, as well as the Priorean temporal language (with forward and backward looking diamonds **F** and **G** and their respective duals **P** and **H**) in the standard way. Such structures naturally support the interpretation of a hierarchy of more expressive languages with modalities corresponding to different combinations and interactions of quantifiers over branches and instants, see e.g. [2].

3 Completeness Results

Some completeness results have been obtained in the modal language (see e.g. ([4]) for finite trees), though there are very few results for Priorean temporal logics on trees. Also, [3] published completeness results for some of the linear classes of frames. Methods from this and other early works can be adapted to get completeness results for some classes of trees, while we had to develop some new methods for other classes.

Completeness results obtained for Priorean temporal logics of various classes of irreflexive and reflexive trees are summarized in Tables 3 and 4, respectively. The axioms used are listed in Table 1. All logics include the standard rules for modus ponens, uniform substitution and **G** and **H** generalization. A few logics require non-standard rules with syntactic side conditions. The latter are listed in Table 2.

4 Continuous Irreflexive Trees

This axiomatisation was built on the axiomatisation of the real numbers done by Segerberg in [3]. Let $\mathbf{A}\phi$ be an abbreviation for $\mathbf{H}\phi \wedge \phi \wedge \mathbf{G}\phi$ and $\mathbf{E}\phi$ an abbrevi-

	Rule	Description
IRR	If $\vdash \neg p \wedge \mathbf{H}p \rightarrow \phi$ then $\vdash \phi$ where $p \notin var(\phi)$.	Irreflexivity Rule
FDR	If $(\vdash \mathbf{F}p \wedge \neg\mathbf{FF}p \rightarrow \phi) \vee \mathbf{G}\bot$ then $\vdash \phi$ where $p \notin var(\phi)$.	Forwards Discrete Rule
BRD	If $(\vdash \mathbf{P}p \wedge \neg\mathbf{PP}p \rightarrow \phi) \vee \mathbf{H}\bot$ then $\vdash \phi$ where $p \notin var(\phi)$.	Backwards Discrete Rule

Table 2
Additional rules of inference for Priorean temporal logics of trees

Logic	**AxiomsandRules**	**Complete for classes of irreflexive trees**	**Strong / Weak**
$\mathbf{Pr}_{basic}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, **4**, $.\mathbf{3_l}$	All	S
$\mathbf{Pr}_{basic}\mathbf{U_l}$	$\mathbf{Pr}_{basic} + \mathbf{U_l}$	Left unbounded	S
$\mathbf{Pr}_{basic}\mathbf{U_r}$	$\mathbf{Pr}_{basic} + \mathbf{U_r}$	Right unbounded	S
$\mathbf{Pr}_{unbnd}$	$\mathbf{Pr}_{basic} + \mathbf{U_l} + \mathbf{U_r}$	Unbounded	S
$\mathbf{Pr}_{dense}$	$\mathbf{Pr}_{basic} + \mathbf{D}$	Dense	S
$\mathbf{Pr}_{dense}\mathbf{U_l}$	$\mathbf{Pr}_{dense} + \mathbf{U_l}$	Left unbounded dense	S
$\mathbf{Pr}_{dense}\mathbf{U_r}$	$\mathbf{Pr}_{dense} + \mathbf{U_r}$	Right unbounded dense	S
$\mathbf{Pr}_{\mathbb{Q}}$	$\mathbf{Pr}_{dense} + \mathbf{U_l} + \mathbf{U_r}$	Branches iso. to $\mathbb{Q}$	S
$\mathbf{Pr}_{ind}$	$\mathbf{Pr}_{basic} + \mathbf{S}$	Locally finite	W
$\mathbf{Pr}_{ind}\mathbf{U_l}$	$\mathbf{Pr}_{ind} + \mathbf{U_l}$	Left unbound locally finite	W
$\mathbf{Pr}_{ind}\mathbf{U_r}$	$\mathbf{Pr}_{ind} + \mathbf{U_r}$	Right unbound locally finite	W
$\mathbf{Pr}_{ind}\mathbf{U_l}\mathbf{U_r}$	$\mathbf{Pr}_{ind} + \mathbf{U_l} + \mathbf{U_r}$	Unbounded locally finite	W
$\mathbf{Pr}_{\mathbb{Z}}$	$\mathbf{Pr}_{basic}$ + **S**, $\mathbf{U_l}$, $\mathbf{U_r}$	Branches iso. to $\mathbb{Z}$	W
$\mathbf{Pr}_{discr}$	$\mathbf{Pr}_{basic}$, IRR, FDR, BDR	Discrete	S
$\mathbf{Pr}_{discr}\mathbf{U_l}$	$\mathbf{Pr}_{discr} + \mathbf{U_l}$	Left unbound discrete	W
$\mathbf{Pr}_{discr}\mathbf{U_r}$	$\mathbf{Pr}_{discr} + \mathbf{U_r}$	Right unbound discrete	W
$\mathbf{Pr}_{udiscr}$	$\mathbf{Pr}_{discr} + \mathbf{U_l} + \mathbf{U_r}$	Unbounded discrete	S
$\mathbf{Pr}_{finite}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, $.\mathbf{3_l}$, $\mathbf{L_l}$, $\mathbf{L_r}$	Finite	W
$\mathbf{Pr}_{well\text{-}fnd}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, $.\mathbf{3_l}$, $\mathbf{L_l}$	Well-founded	W
$\mathbf{Pr}_{cwell\text{-}fnd}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, $.\mathbf{3_l}$, $\mathbf{L_r}$	Conversely well-founded	W
$\mathbf{Pr}_{\mathbb{N}}$	$\mathbf{Pr}_{well\text{-}fnd}$ + **S**, $\mathbf{U_r}$	Branches iso. to $\mathbb{N}$	W

Table 3
Completeness results for Priorean temporal logics of classes of irreflexive trees

ation for $\mathbf{P}\phi \vee \phi \vee \mathbf{F}\phi$ and consider the following two formulas:

- $\mathbf{C}_l$: $\mathbf{A}(\mathbf{P}p \leftrightarrow \mathbf{G}p) \rightarrow (\mathbf{E}p \rightarrow \mathbf{A}p)$
- $\mathbf{C}_r$: $\mathbf{A}(\mathbf{F}p \leftrightarrow \mathbf{H}p) \rightarrow (\mathbf{E}p \rightarrow \mathbf{A}p)$

Lemma 4.1 *$\mathbf{C}_l$ and $\mathbf{C}_r$ are valid on trees with branches isomorphic to $\langle \mathbb{R}, < \rangle$.*

Let $\mathbf{Pr}_{\mathbb{R}}$ denote the logic $\mathbf{K_t4.3}_l\mathbf{DU_lU_rC}_l\mathbf{C}_r$, where $\mathbf{K_t}$ is the standard normal tem-

Logic	Axioms/Rules	Complete for classes of reflexive trees	Strong / Weak
$\mathbf{Pr}_{basic}\mathbf{T}$	$\mathbf{Pr}_{basic} + \mathbf{T}$	All	S
$\mathbf{Pr}_{basic}\mathbf{TU_l}$	$\mathbf{Pr}_{basic} + \mathbf{T}, \mathbf{U_l}$	Left unbounded	S
$\mathbf{Pr}_{basic}\mathbf{TU_r}$	$\mathbf{Pr}_{basic} + \mathbf{T}, \mathbf{U_r}$	Right unbounded	S
$\mathbf{Pr}_{unbnd}\mathbf{T}$	$\mathbf{Pr}_{unbnd} + \mathbf{T}$	Unbounded	S
$\mathbf{Pr}_{dense}\mathbf{T}$	$\mathbf{Pr}_{dense} + \mathbf{T}$	Dense	S
$\mathbf{Pr}_{dense}\mathbf{TU_l}$	$\mathbf{Pr}_{dense} + \mathbf{T}, \mathbf{U_l}$	Left unbounded dense	S
$\mathbf{Pr}_{dense}\mathbf{TU_r}$	$\mathbf{Pr}_{dense} + \mathbf{T} + \mathbf{U_r}$	Right unbounded dense	S
$\mathbf{Pr}_{(\mathbb{Q},\leq)}$	$\mathbf{Pr}_{\mathbb{Q}} + \mathbf{T}$	Unbounded dense; branches isomorphic to $(\mathbb{Q}, \leq)$	S
$\mathbf{Pr}_{finref}$	$\mathbf{K_t}$, $\mathbf{.3_l}$, $\mathbf{Grz}$, $\mathbf{Grz_l}$	Finite	W
$\mathbf{Pr}_{basic}\mathbf{TGrz_l}$	$\mathbf{Pr}_{basic} + \mathbf{T}, \mathbf{Grz_l}$	Well-founded	W
$\mathbf{Pr}_{basic}\mathbf{TGrz}$	$\mathbf{Pr}_{basic} + \mathbf{T}, \mathbf{Grz}$	Conversely well-founded	W
$\mathbf{Pr}_{(\mathbb{N},\leq)}$	$\mathbf{Pr}_{basic} + \mathbf{T}, \mathbf{Grz_l}, \mathbf{U_l}$	Branches iso. to $(\mathbb{N}, \leq)$	W

Table 4
Completeness results for Priorean temporal logics of classes of reflexive trees

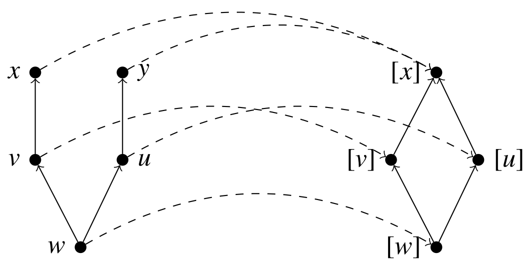

Fig. 1. Example of identification and preserving original relations that is not left-linear

poral logic, and the additional axioms are listed in Table 1. We will sketch a proof that $\mathbf{Pr}_{\mathbb{R}}$ is weakly complete with respect to the class of trees with branches isomorphic to $\langle \mathbb{R}, < \rangle$. Let α be any non-theorem of $\mathbf{Pr}_{\mathbb{R}}$ and let Φ_α be the set of subformulas of α. Let w_0 be an instant in the canonical model for $\mathbf{Pr}_{\mathbb{R}}$ such that $\alpha \notin w_0$ and let $\mathcal{M} = (W, R, V)$ be the submodel of the canonical model generated by w_0. Then $\mathcal{M}$ is a tree of clusters with $\mathcal{M}, w_0 \nVdash \alpha$. We build a finite model for $\mathbf{Pr}_{\mathbb{R}}$ in which α is false as follows. We start with $\mathcal{M} = (W, R, V)$, the submodel of the canonical model of $\mathbf{Pr}_{\mathbb{R}}$ generated by w_0 (where w_0 is a mcs containing $\neg\alpha$) and use transitive filtration [1] to get $\mathcal{M}' = (W', R', V')$ where the R' is defined as follows: For all $[w], [v] \in W'$, $R'[w][v]$ iff (for all $\mathbf{G}\gamma \in \Phi_\alpha$ with $\mathbf{G}\gamma \in w$ we have $\mathbf{G}\gamma, \gamma \in v$) and (for all $\mathbf{H}\gamma \in \Phi_\alpha$ with $\mathbf{H}\gamma \in v$ we have $\mathbf{H}\gamma, \gamma \in w$). The filtration produces a finite, transitive and connected model. However, $\mathcal{M}'$ is not necessarily left-linear. The left linearity of the generated submodel $\mathcal{M}$ can be lost in two different ways, one of which shown in Figure 1, and the other looks similarly.

Lemma 4.2 *The filtration $\mathcal{M}'$ has a single root cluster.*

So, the filtration produces a finite, transitive, connected frame with a single root cluster. The next step in building the required model is to remove all left-linear defects. We work on the filtration-quotient frame to get a tree of clusters. Through **refining the filtration** we remove some of the unnecessary relation pairs and prepare the model for removing the remaining left-linearity defects. This process results into a model $\mathcal{M}^r$ that also satisfies the conditions of a filtration and hence gives us the following:

Lemma 4.3 *For all $\phi \in \Phi_\alpha$ and for all $w \in W$ we have $\mathcal{M}, w \Vdash \phi$ iff $\mathcal{M}^r, [w] \Vdash \phi$.*

Our next goal is to repair the remaining left-linearity defects. For this purpose we use **unfolding** to build a model $\mathcal{M}''$ that is again a tree of clusters. We also show that, on the set Φ_α, $\mathcal{M}''$ satisfies the same formulas as $\mathcal{M}^r$ at corresponding clusters and that $\mathcal{M}''$ is a finite tree of clusters with non-degenerate cluster at the root and leaves, such that $\mathcal{M}'', w_0 \nVdash \alpha$. Investigating the structure of $\mathcal{M}''$ further, one can see that no degenerate clusters are adjacent, and if two non-degenerate clusters are adjacent, we can isolate an irreflexive instant "between" these to clusters to turn $\mathcal{M}''$ into a model $\mathcal{M}^*$ that has alternating degenerate and non-degenerate clusters. Therefore, $\mathcal{M}^*$ is a finite tree of clusters with no two adjacent degenerate clusters and between any non-degenerate clusters there is a single irreflexive instant (degenerate cluster). It also has a non-degenerate cluster as root and non-degenerate clusters at the leaves.

We are now ready to turn this model into a tree with branches isomorphic to the real line. By **unrolling** the non-degenerate clusters into strict linear orderings isomorphic to the reals, we can turn every non-degenerate cluster into such a linear ordering and the resulting model $\mathcal{M}^{**}$ is a tree. Since the root and leaves of $\mathcal{M}^*$ are non-degenerate and there are no adjacent degenerate clusters, it follows that $\mathcal{M}^{**}$ will have unrolled clusters at the root and leaves, and on each branch the clusters will alternate between degenerate and unrolled non-degenerate clusters. Hence, the branches of the model $\mathcal{M}^{**}$ are strictly linearly ordered. Therefore $\mathcal{M}^{**}$ is an irreflexive tree. Next, we show that g is a bounded morphism. Furthermore, there is a $w'_0 \in W^{**}$ such that $g(w'_0) = [w_0]$, hence $\mathcal{M}^{**}, w'_0 \nVdash \alpha$. Thus, we can define an isomorphism $h : H \to \mathbb{R}$, where H is a branch in $\mathcal{M}^{**}$, which gives the following completeness result.

Theorem 4.4 $\mathbf{Pr}_{\mathbb{R}}$ *is weakly complete with respect to the class of all trees with branches isomorphic to $\langle \mathbb{R}, < \rangle$.*

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Characterising Modal Formulas with Examples

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Abstract

We initiate the study of finite characterizations and exact learnability of modal languages. A finite characterization of a modal formula w.r.t. a set of formulas is a finite set of finite models (labelled either positive or negative) which distinguishes this formula from every other formula from that set. A modal language $\mathcal{L}$ admits finite characterisations if every $\mathcal{L}$-formula has a finite characterization w.r.t. $\mathcal{L}$. This definition can be applied not only to the basic modal logic **K**, but to arbitrary normal modal logics. We show that a normal modal logic admits finite characterisations (for the full modal language) iff it is locally tabular. This shows that finite characterizations with respect to the full modal language are rare, and hence motivates the study of finite characterizations for fragments of the full modal language. Our main result is that the positive modal language without the truth-constants $\top$ and $\bot$ admits finite characterisations. Moreover, we show that this result is essentially optimal: finite characterizations no longer exist when the language is extended with the truth constant $\bot$ or with all but very limited forms of negation.

Keywords: Finite Characterisations, Positive Languages, Exact Learning, Simulation, Dualities, Description Logic

1 Introduction

Every modal formula defines a possibly infinite set of pointed models that satisfy it (*positive* examples), and implicitly also the set of pointed models that do not satisfy it (*negative* examples). We study the question whether for a given class of modal formulas it its possible to *characterize* every formula with a *finite* set of positive and negative examples such that that no other formula is consistent with these. We call such sets of examples *finite characterisations.* The existence of such finite characterizations is a precondition for the existence *exact learning algorithms* for 'reverse-engineering' a hidden goal formula from

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examples in Angluin's model of exact learning with membership queries [2]. Our interest in exact learnability is motivated by applications in description logic. But besides learnability, the generation of examples consistent with a given formula is of interest in query visualization and explanation, and is closely connected to the data separability problem.

In this extended abstract, we only provide a high level description of our results and proof techniques. Detailed proofs can be found here: `https://bit.ly/3LCtmQt`.

2 Preliminaries

Given a set of propositional variables Prop and a set of connectives $C \subseteq \{\wedge, \vee, \Diamond, \Box, \top, \bot\}$, let $\mathcal{L}_C[\text{Prop}]$ (or simply $\mathcal{L}_C$ when Prop is clear from context) denote the collection of all modal formulas generated from literals (i.e. positive or negated propositional variables) from Prop, using the connectives in C. Note that all such formulas are in negation normal form, i.e. negations may only occur in front of propositional variables. Further, for any modal fragment $\mathcal{L}$ as defined above, $\mathcal{L}^+$ and $\mathcal{L}^-$ denote the set of positive, respectively negative $\mathcal{L}$ formulas, where a formula φ is *positive* if no $p \in var(\varphi)$ occurs negated, and *negative* if all $p \in var(\varphi)$ occur only negated. We will use *modal language* to refer to any such fragment. By the *full modal language* we will mean $\mathcal{L}_{\Box,\Diamond,\wedge,\vee,\top,\bot}[\text{Prop}]$.

For a modal formula φ, let $var(\varphi)$ denote the set of variables occurring in φ and $d(\varphi)$ its *modal depth*, i.e. the nesting depth of $\Diamond$'s and $\Box$'s in φ.

A *normal modal logic* is a collection of modal formulas containing all instances of the K-axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and closed under uniform substitution, modus ponens and generalisation. A (Kripke) *model* is a triple $M = (dom(M), R, v)$ where $dom(M)$ is the a set of 'possible worlds', $R \subseteq dom(M) \times dom(M)$ a binary 'accessibility' relation and a valuation $V : \text{Prop} \to \mathcal{P}(W)$. A *pointed model* is a pair M, s of a Kripke model M together with a state $s \in dom(M)$. A (Kripke) *frame* is a model without its valuation.

3 Finite Characterizations

First, we give a formal definition of finite characterizations of modal formulas.

Definition 3.1 (Finite characterizations) A *finite characterization* of a formula $\varphi \in \mathcal{L}[\text{Prop}]$ w.r.t. $\mathcal{L}[\text{Prop}]$ is a pair of finite sets of finite pointed models $\mathbb{E} = (E^+, E^-)$ such that (i) φ *fits* (E^+, E^-), i.e. $E, e \models \varphi$ for all $(E, e) \in E^+$ and $E, e \not\models \varphi$ for all $(E, e) \in E^-$ and (ii) φ is the only formula in $\mathcal{L}[\text{Prop}]$ which fits (E^+, E^-), i.e. if $\psi \in \mathcal{L}[\text{Prop}]$ satisfies condition (i) then $\varphi \equiv \psi$. A modal language $\mathcal{L}$ is *finitely characterizable* if for every finite set of propositional variables Prop, every $\varphi \in \mathcal{L}[\text{Prop}]$ with has a finite characterization w.r.t. $\mathcal{L}[\text{Prop}]$.

Thus if (E^+, E^-) is a finite characterization of a formula $\varphi \in \mathcal{L}[\text{Prop}]$ w.r.t. $\mathcal{L}[\text{Prop}]$, then for every $\psi \in \mathcal{L}[\text{Prop}]$ with $\varphi \not\equiv \psi$, E^+ contains a finite model of $\varphi \wedge \neg\psi$ or E^- contains a finite model of $\neg\varphi \wedge \psi$. For example, the formula $p \wedge q$ has a finite characterization w.r.t. $\mathcal{L}^+_\wedge[\text{Prop}]$ with Prop $= \{p, q, r\}$, namely $(\{\cdot_{p,q}\}, \{\cdot_p, \cdot_q\})$, where "$\cdot_P$" is the single point model where all $p \in P$ are true.

Our motivation for studying finite characterizations, comes from *computational learning theory*. Specifically, finite characterizability is a necessary precondition for *exact learnability with membership queries* in Dana Angluin's interactive model of exact learning [2]. In our context, exact learnability with membership corresponds to a setting in which the learner has to identify a formula by asking question to an oracle, where each question is of the form "is the formula true or false in pointed model (M, w)?" This can also be viewed as a 'reverse engineering' task, where a formula has to be identified based on its behaviour on only a finite set of models. Exact learnability has recently gained a renewed interest in the description logic literature. We comment more on the connection with description logic in Section 4.

Our starting observation is:

Theorem 3.2 *The full modal language is not finitely characterizable.*

Proof. It suffices to give one counterexample, so suppose that e.g. $\varphi = \Box\bot$ had a finite characterization (E^+, E^-) w.r.t. the full modal language. Observe that for each n, $M, s \models \Box^{n+1}\bot \wedge \Diamond^n\top$ iff *height*$(M, s) = n$, where the *height* of a pointed model M, s is the length of the longest path in M starting at s, or ∞ if there is no finite upper bound. Every finite characterizations can only contain pointed models up to some bounded height. Hence for large enough n, $\varphi \vee (\Box^{n+1}\bot \wedge \Diamond^n\top)$ must also fit (E^+, E^-), yet is clearly not equivalent to φ.$\Box$

In fact, by a variation of the same argument, we can show that *no* modal formula has a finite characterization w.r.t. the full modal language.

Theorem 3.2 raises two questions, namely: *do finite characterizations exist in other modal logics than* **K**, and *which fragments of modal logic admit finite characterizations*. We address each of these two questions next.

We first generalize Definition 3.1 as follows (whereby Theorem 3.2 becomes a result about the special case of the basic normal modal logic **K**): a finite characterization of a modal formula φ with $var(\varphi) \subseteq$ Prop w.r.t. a normal modal logic L is a finite set (E^+, E^-) of finite pointed models based on L frames such that (i) φ fits (E^+, E^-) and (ii) if ψ with $var(\psi) \subseteq$ Prop fits (E^+, E^-) then $\varphi \equiv_L \psi$, where $\varphi \equiv_L \psi$ iff $\varphi \leftrightarrow \psi \in L$. We say that a normal modal logic L is finitely characterizable if for every finite set Prop, every modal φ with $var(\varphi) \subseteq$ Prop has a finite characterization w.r.t. L. We can give a complete characterization over which modally definable frame classes the full modal language is finitely characterizable.

It turns out that only very few normal modal logics are uniquely characterizable. A normal modal logic L is *locally tabular* if for every finite set Prop of propositional variables, there are only finitely many formulas φ with $var(\varphi) \subseteq$ Prop up to L-equivalence.

Theorem 3.3 *A normal modal logic L is finitely characterizable iff it is locally tabular.*

In other words the full modal language is only finitely characterizable in the degenerate case where there are only finitely many formulas to distinguish from (up to equivalence). This result motivates the investigation of finite characterizability for modal fragments. Specifically, inspired by previous work on finite characterizability of the positive existential fragment of first order logic [5], we consider positive fragments of the full modal language.

In the remainder of this section, we only consider again the modal logic **K**. The proof of Theorem 3.2 can easily be modified to show the following: [3]

Theorem 3.4 *$\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee,\bot}$ is not finitely characterizable.*

On the other hand, based on results in [5], we can show that:

Theorem 3.5 (From [5]) *$\mathcal{L}^+_{\Diamond,\wedge}$ is finitely characterizable. Indeed, given a formula in $\mathcal{L}^+_{\Diamond,\wedge}$, we can construct a finite characterization in polynomial time.*

More precisely, it was shown in [5] that finite characterizations can be constructed in polynomial time for "c-acyclic conjunctive queries", a fragment of first-order logic that includes the standard translations of $\mathcal{L}^+_{\Diamond,\wedge}$-formulas.

Our main result here extends Theorem 3.5 by showing that $\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee}$ is finitely characterizable.

Theorem 3.6 *$\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee}$ is finitely characterizable.*

Theorem 3.4 above shows that this is essentially optimal; we leave open the question whether the fragment without $\bot$ but with $\top$ is finitely characterizable.

In the rest of this section, we outline the ideas behind the proof of Theorem 3.6. A key ingredient is the novel notion of *weak simulation*, which we obtain by weakening the back and forth clauses of the *simulations* studied in [4]. Simulations are themselves a weakening of bisimulations. It was shown in [4] that $\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee,\top,\bot}$ is characterized by preservation under simulations.

A *weak simulation* between two pointed models $(M, s), (M', s')$ is a relation $Z \subseteq M \times M'$ such that for all $(t, t') \in Z$:

(atom) $M, s \models p$ implies $M', s' \models p$
(forth′) If $R^M tu$, either $M, u \underline{\leftrightarrow} \circlearrowright_\emptyset$ or there is a u' with $R^{M'} t'u'$ and $(u, u') \in Z$
(back′) If $R^{M'} t'u'$, either $M', u' \underline{\leftrightarrow} \circlearrowright_{\mathrm{Prop}}$ or there is a u with $R^M tu$ and $(u, u') \in Z$

where $\circlearrowright_\emptyset$ denotes the single reflexive point with empty valuation, $\circlearrowright_{\mathrm{Prop}}$ denotes the single reflexive point with full valuation and $\underline{\leftrightarrow}$ denotes bisimulation. If such Z exists, we say that M', s' *weakly simulates* M, s. The crucial weakening is witnessed by the fact that the deadlock model, i.e. the single point with no successors, weakly simulates $\circlearrowright_\emptyset$, but does not simulate it.

Because weak simulations are closed under relational composition, which is associative, the collection of pointed models and weak simulations forms a category with $\circlearrowright_\emptyset$ and $\circlearrowright_{\mathrm{Prop}}$ as weak initial and final objects, respectively.

Theorem 3.7 *$\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee}$ is preserved under weak simulations.*

[3] It suffices to replace $\top$ by a fresh propositional variable q in the proof of Theorem 3.2.

In high level terms, the proof of Theorem 3.6 proceeds as follows: given a formula $\varphi \in \mathcal{L}^{+}_{\Box,\Diamond,\wedge,\vee}$, we show how to construct positive and negative examples $(E^{+}_{\varphi}, E^{-}_{\varphi})$ that φ fits and which forms a *duality* (a generalised form of splittings for categories) in the in the category of pointed models and weak simulations. More specifically, we show that every model of φ weakly simulates some positive example in E^{+} and that every non-model of φ is weakly simulated by some negative example in E^{-}. It follows by Theorem 3.7 that any $\mathcal{L}^{+}_{\Box,\Diamond,\wedge,\vee}$-formula that fits E^{+} is implied by φ, while every formula that fits E^{-} implies φ, showing that $(E^{+}_{\varphi}, E^{-}_{\varphi})$ is a finite characterization of φ w.r.t. $\mathcal{L}^{+}_{\Box,\Diamond,\wedge,\vee}$.

This proof technique was inspired by results in [1], which established a similar connection between finite characterizations for GAV schema mappings (or, equivalently, unions of conjunctive queries) and homomorphism dualities (i.e. generalised splittings in the category of finite structures and homomorphisms. See `https://bit.ly/3LCtmQt` for more details and further results.

4 Discussion

Our construction, although effective, is non-elementary. For this reason, we cannot obtain from it an efficient exact learning algorithm. On the other hand, it follows from the results in [5] that $\mathcal{L}^{+}_{\Diamond,\wedge}$-formulas are polynomial-time exactly learnable with membership queries. We leave it as future work to prove matching lower bounds for our construction, and to understand more precisely which modal fragments admit polynomial-sized finite characterizations and/or are polynomial-time exactly learnable with membership queries.

Variants of Theorem 3.6 can be obtained for $\mathcal{L}^{-}_{\Box,\Diamond,\wedge,\vee}$ and, more generally, for *uniform* modal formulas, where certain propositional variables only occur positive and others only negatively.

As we mentioned, our immediate motivation for this work came from a renewed interest in exact learnability in description logic. In particular, in [5,3], exact learnability with membership queries is studied for the description logic $\mathcal{ELI}$ in the presence of DL-Lite ontologies (i.e. background theory) Therefore, it of interest to generalize our results to the poly-modal case, with backward modalities, and in the presence of an ontology. We expect that our proof of Theorem 3.6 can be lifted to the poly-modal case without major changes.

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Strict Implication Over Logics Without Contraction

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Abstract

This study provides an algebraic investigation of strict implications defined over substructural base logics, in particular those lacking the contraction rule of structural proof theory. Our main contribution consists of the identification of a large class of residuated lattices that may be embedded in so-called poset products of simple residuated lattices, providing a substructural analogue of the well-known embedding of each Heyting algebra into the $\Box$-fixed elements of a corresponding interior algebra. We further consider an infinite family of varieties of residuated lattices that are subsumed by our framework, giving infinitely-many corresponding substructural logics whose implication connective is a strict implication over a substructural base logic.

Keywords: Strict implication, substructural logics, residuated lattices, poset products, translations.

In classical modal logic, *strict implication* typically refers to a derived connective arising from prefixing the material implication $\to$ by a modal necessity operator $\Box$, i.e., the derived logical connective $(p, q) \mapsto \Box(p \to q)$. Strict implication has been an important topic in modal logic since its modern beginnings [10] (see also [11]), where it has been deployed to resolve paradoxes of material implication. For any given modal logic extending $\mathsf{S4}$, this connective is well-known to correspond to the implication of a superintuitionistic logic (see, e.g., [1,4]). The present work concerns strict implication defined over non-classical base logics. In particular, we offer an algebraic study of strict implication logics that arise from Kripke frames (in the sense of [5]) whose worlds are valued

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in algebraic models of logics lacking the contraction rule of structural proof theory (see, e.g., [12]). This generalizes the classical case, where we may understand worlds as being valued in the 2-element Boolean algebra. Our main contribution consists of the identification of a large class of (bounded, integral, commutative) residuated lattices [7] that embed in strict implication algebras that arise from the aforementioned frames. Although this class is not even a quasivariety and thus does not directly correspond to a logic, we identify several natural varieties contained in this class. As a consequence, we obtain axiomatizations of several natural substructural logics whose implication is a strict implication in the previously articulated sense.

1 Background and key definitions

Recall that a (bounded, integral, commutative) *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 0, 1)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \cdot, 1)$ is a commutative monoid, and for all $a, b, c \in A$,

$$a \cdot b \leq c \iff a \leq b \to c.$$

Residuated lattices give the equivalent algebraic semantics for the Full Lambek calculus with exchange, weakening, and falsity, but possibly lacking the contraction rule. In any residuated lattice $\mathbf{A}$, we may consider its *idempotent center*:

$$\mathcal{H}(A) = \{a \in A : a^2 = a\}.$$

The idempotent center of a residuated lattice is not in general a subalgebra, but this occurs in several natural cases. For instance, Heyting algebras are precisely those residuated lattices $\mathbf{A}$ for which $A = \mathcal{H}(A)$. Our key technical contributions concern residuated lattices for which $\mathcal{H}(A)$ is situated inside in an especially nice way. The following defines the main class of interest.

Definition 1.1 A residuated lattice $\mathbf{A}$ is called *centered* if:

(i) $\mathcal{H}(A)$ is a subalgebra of $\mathbf{A}$.

(ii) For every $i \in \mathcal{H}(A)$ and $a \in A$, there exists $j \in \mathcal{H}(A)$ such that $i \wedge j \leq a \leq i \vee j$.

(iii) Every filter of $\mathbf{A}$ is generated by the idempotent elements it contains.

The importance of centered residuated lattices to strict implication comes from their amenability to analysis by *poset products* [9,8]. Poset products were introduced as a common generalization of direct products and ordinal sums. To describe the poset product construction in more detail, let $(X, \leq)$ be poset and let $\{\mathbf{A}_x : x \in X\}$ be an indexed collection of residuated lattices with common bottom 0 and top 1. Set

$$\mathbf{B} = \prod_{x \in X} \mathbf{A}_x$$

and define a map $\Box\colon B \to B$ by

$$\Box(f)(x) = \begin{cases} f(x) & \text{if } f(y) = 1 \text{ for all } y > x \\ 0 & \text{if there exists } y > x \text{ with } f(y) \neq 1. \end{cases}$$

The set of fixed points $B_\Box = \{f \in B : \Box f = f\}$ forms a residuated lattice, where $\wedge, \vee, \cdot, 0, 1$ are interpreted as in the direct product and $\to$ is given by prefixing the direct product implication by $\Box$. The resulting residuated lattice $\mathbf{B}_\Box$ is called the *poset product* of the indexed family of residuated lattices, and is denoted by

$$\prod_{(X,\leq)} \mathbf{A}_x.$$

The operation $\Box$ is a decreasing, idempotent, and monotone operator, and generalizes the modal operator native to interior algebras (see [5] for an interpretation in terms of $\mathsf{S4}$-like Kripke frames, and see [6] for an interpretation in terms of Gödel-style translations). Commensurately, the implication operation associated with poset products is an algebraic interpretation of a strict implication over the logic algebraized by the factors $\mathbf{A}_x$, $x \in X$.

2 Main results

Recall that an algebra is *simple* if it has no non-trivial congruences. An example of a simple residuated lattice is given by the 2-element Boolean algebra. Our main technical result, generalizing many of the main results of [9], is as follows.

Theorem 2.1 *Every centered residuated lattice embeds into a poset product of simple residuated lattices.*

In the case where each factor of the poset product is the 2-element Boolean algebra, this reduces to the well-known result for Heyting algebras and interior algebras: Each Heyting algebra embeds as a subalgebra of the $\Box$-fixed elements of some interior algebra. On a conceptual level, the previous theorem realizes each centered residuated lattice as one for which the operation interpreting implication comes from the strict implication of some Kripke frame whose worlds are valued in simple residuated lattices.

The proof of Theorem 2.1 rests on a generalization of the celebrated Blok-Ferreirim theorem [2,8] for centered residuated lattices. While this result is primarily of interest as a technical lemma in our work, the result provides significant insight into congruences of centered residuated lattices and is worth stating in its own right:

Theorem 2.2 *Let* $\mathbf{A}$ *be a subdirectly irreducible centered residuated lattice. Then there is a maximum element* m *of* $\mathcal{H}(A) \setminus \{1\}$*, and for all* $a \in A$ *we have* $m \leq a$ *or* $a \leq m$*.*

3 Varieties and logics

The class of centered residuated lattices is evidently not a quasivariety, and hence does not correspond directly to a deductive system under the usual apparatus of abstract algebraic logic. However, the class of centered residuated lattices is sufficiently large to encompass an array of varieties. These each comprise a substructural logic whose implication may, following Theorem 2.1, be interpreted as a logic of strict implication.

Definition 3.1 For each $n \in \mathbb{N}$, denote by S_n the subvariety of residuated lattices axiomatized by:

(i) $a^n b = a^n \wedge b$.

(ii) $a^n \to b^n = (a^n \to b^n)^2$.

(iii) $a \leq b^n \vee (b^n \to a^n)$.

Further, for each $n \in \mathbb{N}$ denote by C_n the subvariety of S_n axiomatized by

(iv) $(a \to b) \to (b \to a) = b \to a$.

A residuated lattice $\mathbf{A}$ is called *n-potent* if it satisfies $a^{n+1} = a^n$ for all $a \in A$, and it is called an *MTL-algebra* if it satisfies $(a \to b) \vee (b \to a) = 1$ for all $a, b \in A$. Equivalently, MTL-algebras are precisely the residuated lattices in the variety generated by totally ordered residuated lattices. Note that given $n \in \mathbb{N}$, the variety C_n contains the variety of Heyting algebras, as well as the variety of n-potent bounded commutative GBL-algebras (see, e.g., [9]), and therefore also the variety of n-potent MV-algebras (which are MTL-algebras). By an application of Theorem 2.1, we obtain the following:

Theorem 3.2 *Let $n \in \mathbb{N}$.*

(i) *The variety S_n is generated by the class of poset products of simple n-potent residuated lattices.*

(ii) *The variety C_n is generated by the class of poset products of simple n-potent MTL-algebras.* [3]

Because each variety of residuated lattices corresponds to an axiomatic extension of the Full Lambek calculus, Theorem 3.2 provides an infinite family of substructural logics whose implication connective is a strict implication. In each of these cases, the underlying base logic is a substructural logic without contraction which is sound and complete with respect to a class of simple algebraic models.

The ramifications of these structural results is far-reaching, both in algebraic and logical terms. By applying the tools developed in [5,6], the previously-announced poset product representations may be used to provide Kripke-style semantics and modal translations for the introduced logics. Future work will focus on deepending these results, providing a more complete theory of strict

[3] Note that in [3], finite simple MTL-algebras are called *archimedean*.

implication over substuctural bases and clarifying the role of the emerging field of substructural modal logics.

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Some new examples of Kripke complete modal predicate logics

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Abstract

The paper gives an overview of recent results on completeness of modal predicate logics in Kripke semantics obtained by applying canonical models and their modifications.

Keywords: modal predicate logic, Kripke completeness, canonical model

1 Introduction and preliminaries

Since the 1990s it has been known that many modal predicate logics are incomplete in standard Kripke semantics, unlike propositional logics [1]. Still there are some general completeness theorems for logics with constant domains (cf. [4]), but we do not have any results of this kind for logic with expanding domains. This paper focuses mainly on logics of the form $\mathbf{Q}\Lambda$, minimal predicate extensions of propositional logics Λ. The completeness problem is nontrivial already in this case.

As in [4], we deal with monomodal predicate formulas in the language with countably many predicate letters of all arities ≥ 0, but without equality. A *modal predicate logic* is a set of formulas containing $\mathbf{K}$, classical predicate axioms and closed under Modus Ponens, Generalization ($A/\forall xA$), Necessitation ($A/\Box A$) and predicate Subsitution.

$\mathbf{Q}\Lambda$ denotes the minimal predicate extension of a propositional logic Λ.

A predicate Kripke frame over a propositional frame $F = (W, R)$ is a pair $\mathbf{F} = (F, D)$, where $D = (D_u)_{u \in W}$, with all D_u nonempty and $D_u \subseteq D_v$ whenever uRv. A *valuation* ξ on $\mathbf{F}$ is a function sending each n-ary predicate

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letter P^n_k to a family of n-ary relations: $\xi(P^n_k) = (\xi_u(P^n_k))_{u\in W}$, where $\xi_u(P^n_k) \subseteq D^n_u$. The pair $M = (\mathbf{F}, \xi)$ is a *Kripke model* over $\mathbf{F}$.

The truth of a D_u-sentence (a closed formula with constants from D_u) at a point u from M is defined by recursion in a standard way. In particular,

$$M, u \vDash P^n_k(a_1, \dots, a_n) \Leftrightarrow (a_1, \dots, a_n) \in \xi_u(P^n_k),$$
$$M, u \vDash P^0_k \Leftrightarrow \xi_u(P^0_k) = 1,$$
$$M, u \vDash \forall x A(x) \Leftrightarrow \forall a \in D_u\ M, u \vDash A(a),$$
$$M, u \vDash \Box A \Leftrightarrow \forall v \in R(u)\ M, v \vDash A.$$

A formula $A(\mathbf{x})$ is *valid* at $\mathbf{F}, u$ (in symbols, $\mathbf{F}, u \vDash A$) if its universal closure $\forall \mathbf{x} A(\mathbf{x})$ is true at M, u for any Kripke model M over $\mathbf{F}$; $A(\mathbf{x})$ is valid on $\mathbf{F}$ (in symbols, $\mathbf{F} \vDash A$) if $\mathbf{F}, u \vDash A$ for any point u from $\mathbf{F}$.

The modal logic of a class of frames $\mathcal{C}$ is the set $\mathbf{L}(\mathcal{C})$ of all formulas valid on all frames from $\mathcal{C}$. Logics of this kind are called *Kripke complete.*

2 Canonical logics

Recall the definition of canonical models from [4].

Definition 2.1 Fix a countable set of constants S^*. An *L-place* is a maximal L-consistent Henkin theory with constants from some co-infinite ('small') subset of S^*.

The *canonical mdoel* M_L of a modal predicate logic L has the form (W_L, R_L, D_L, ξ_L), where

- W_L is the set of all L-places,
- $\Gamma R_L \Delta$ if $\Box A \in \Gamma$ implies $A \in \Delta$,
- $(D_L)_\Gamma$ (abbreviated to D_Γ) is the set of all constants appearing in Γ,
- $M_L, \Gamma \vDash A \Leftrightarrow A \in \Gamma$ for atomic D_Γ-sentences A.

Theorem 2.2 *[Canonical model theorem] For any L-place Γ and D_Γ-sentence A, $\quad M_L, \Gamma \vDash A \Leftrightarrow A \in \Gamma$.*

Definition 2.3 A modal predicate logic L is *canonical* if $\mathbf{F}_L \vDash L$

Corollary 2.4 *Every canonical logic is Kripke complete.*

Lemma 2.5 *Let L and L' be modal predicate logics such that $L \subseteq L'$ and L is canonical. Then $\mathbf{F}_{L'} \vDash L$.*

For the proof cf. [4], Lemma 6.1.25. This lemma readily implies

Proposition 2.6 *The sum of canonical logics is canonical.*

Definition 2.7 A $\mathbf{QK}$-place Γ *verifies* a modal formula A if it contains every D_Γ-sentence, which is a substitution instance of A.

A formula A is called *locally canonical* if for any $\mathbf{QK}$-place Γ verifying A, $\mathbf{F}_{\mathbf{QK}}, \Gamma \vDash A$.

Then we have

Proposition 2.8 *If A is a locally canonical formula and the logic $\mathbf{QK} + A$ is is consistent, then it is canonical.*

Lemma 2.9 *Every closed propositional formula is locally canonical.*

Definition 2.10 A *one-way pseudo-transitive formula* is a propositional formula the form $\Box p \to \Box^n p$, where $n \geq 0$, p is a proposition letter.

Lemma 2.11 *Every one-way pseudo-transitive formula is locally canonical.*

Theorem 2.12 *Let A, B be locally canonical formulas without common predicate letters. Then $A \vee B$ is locally canonical.*

The proof follows from the observation that in this case Γ verifies $A \vee B$ iff Γ verifies A or B.

Corollary 2.13 *If A is a locally canonical formula, $\mathbf{QK} + A$ is consistent and B is a closed propositional formula, then $B \to A$ is canonical.*

3 Quasi-canonical logics

Canonical logics are rare; all our examples of the form $\mathbf{Q}\Lambda$ are described by theorems of the previous section. Taking submodels of canonical models is more productive for completeness proofs.

Definition 3.1 A Kripke model $M' = (W', R', D', \xi')$ is a *selective weak submodel* of a Kripke model $M = (W, R, D, \xi)$ if

- $W' \subseteq W$, $R' \subseteq R$,
- for every $w \in W'$, both $D_w = D'_w$ and $\xi'_w = \xi_w$,
- for any $w \in W'$ and D_w-sentence A,

$$M, w \models \Diamond A \Longrightarrow \exists u \in R'(w)\, M, u \models A.$$

Lemma 3.2 *Let $M' = (W', R', D', \xi')$ be a selective weak submodel of $M = (W, R, D, \xi)$. Then, for any $w \in W'$ and D_w-sentence A,*

$$M, w \models A \text{ iff } M', w \models A.$$

Definition 3.3 Let L be a first-order modal logic. A *quasi-canonical model* for L is a selective weak submodel of a canonical model for L.

So from Theorem 3.2 and Lemma 3.2 we obtain

Proposition 3.4 *Let L be a first-order modal logic and $M' = (W', R', D', \xi')$ a quasi-canonical model for L. Then, for every $\Gamma \in W'$,*

$$M', \Gamma \models A \iff A \in \Gamma.$$

Definition 3.5 A modal predicate logic L is *quasi-canonical* if, for every L-place Γ, there exists a quasi-canonical model over a frame validating L and containing Γ.

Theorem 3.6 *Every quasi-canonical modal predicate logic is Kripke complete.*

Typical examples of quasi-canonical logics are $\mathbf{QAlt}_n$ for $n \geq 1$. Recall that $\mathbf{Alt}_n = \mathbf{K} + alt_n$, where

$$alt_n = \neg \bigwedge_{0 \leqslant i \leqslant n} \Diamond (p_i \ \wedge \bigwedge_{j \neq i} \neg p_j).$$

It is well-known that for a propositional frame (W, R),

$$(W, R) \vDash alt_n \text{ iff } \forall u \in W \, |R(u)| \leq n.$$

Theorem 3.7 *The logics* $\mathbf{QAlt}_n$ *are quasi-canonical.*

For the proof cf. [3].

The latter theorem admits further modification.

Definition 3.8 A propositional modal formula A is *locally universal* if there exists a universal classical first-order formula $\alpha(x)$ such that for any frame (W, R) and $u \in W$,

$$(W, R), u \vDash A \text{ iff } (W, R) \vDash \alpha(u) \text{ (classically) .}$$

Theorem 3.9 *Let B, C be propositional formulas such that C is closed, B is locally canonical, locally universal, and does not have common proposition letters with alt_n. Then the logic $\mathbf{QK} + alt_n \vee B \vee C$ is quasi-canonical.*

Definition 3.10 A *path* of length n is a frame (W, R) is a finite sequence of R-related worlds: $w_0 R w_1 \ldots R w_n$. The *depth* of a world u in a frame F (notation: $d(u)$) is the maximal length of a path from u in F if it exists and ∞ otherwise.

Then

$$d(u) = n \text{ iff } F, u \vDash \Box^{n+1} \bot \wedge \Diamond^n \top.$$

We denote the latter formula by dep_n.

Similarly to Theorem 3.9, we have

Theorem 3.11 *If Λ axiomatized by formulas of the form $dep_n \to alt_m$, then $\mathbf{Q}\Lambda$ is quasi-canonical.*

Recall another standard definition:

Definition 3.12 A *tree* is a propositional frame with a root u_0, in which for any world w, there exists a unique path from u_0 to w.

A *uniform tree* of type $(n_1, \ldots, n_k)$ (in symblols, $UT(n_1, \ldots, n_k)$) is a finite tree of depth k, in which

$$|R(w)| = n_i \text{ iff } d(w) = i.$$

Theorem 3.13 *Let $\mathcal{K}UT(n_1, \ldots, n_k)$ be the class of all predicate frames of the form $(UT(n_1, \ldots, n_k), D)$. Then*

$$\mathbf{L}(\mathcal{K}UT(n_1, \ldots, n_k) = \mathbf{QK} + \{dep_i \to alt_{n_i} \mid 1 \leq i \leq k\}.$$

The latter theorem includes the cases when $n_i = \infty$; then we should put $alt_\infty = \top$.

4 Selective filtration

Quasi-canonicity is still exceptional for Kripke complete logics. The most efficient completeness proofs are obtained by selective filtration from canonical models ('natural models', in terms of [4]).

Definition 4.1 Let $M = (W, R, D, \xi)$, $M' = (W', R', D', \xi')$ be predicate Kripke models. A map $h : W' \longrightarrow W$ is a *selective filtration* of M if

- $D'_u = D_{h(u)}$ for any $u \in W'$,
- $M, u \vDash A$ iff $M', h(u) \vDash A$ for any $u \in W'$ and any atomic D_u-sentence A,
- h is monotonic: $uR'v \Rightarrow h(u)Rh(v)$,
- h is selective: for any $u \in W'$ and D_u-sentence A

$$M, h(u) \vDash \Diamond A \Rightarrow \exists v \in R'(u)\, M, h(v) \vDash A.$$

So selective submodels are a particular case of selective filtration, in which h is injective.

Lemma 4.2 *Let $M = (W, R, D, \xi)$, $M' = (W', R', D', \xi')$ be predicate Kripke models, $h : M' \longrightarrow M$ a selective filtration. Then for any $w \in W'$ and D_w-sentence A,*

$$M', w \models A \textit{ iff } M, h(w) \models A.$$

We give two new applications of this method. Consider the following propositional formulas and logics:

$$Ath := \Diamond\Diamond p \to \Box\Diamond p,\;\; A2 := \Diamond\Box p \to \Box\Diamond p.$$

$$\mathbf{K05} := \mathbf{K} + Ath,\;\; \mathbf{K2} := \mathbf{K} + A2.$$

The frame conditions for Ath and $A2$ are respectively *thickness*

$$R^{-1} \circ R^2 \subseteq R$$

and *confluence*

$$R^{-1} \circ R \subseteq R \circ R^{-1}.$$

The logics **QK05** and **QK2** are probably Kripke incomplete. However, by selective filtration we can prove completeness for their relativization to a certain finite depth:

Theorem 4.3 *The logics $\mathbf{Q}(\mathbf{K}+dep_n \to Ath)$, $\mathbf{Q}(\mathbf{K}+dep_n \to A2)$ are Kripke complete.*

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Reasoning with probabilities and belief functions over Belnap–Dunn logic

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Abstract

We design an expansion of Belnap–Dunn logic with belief and plausibility functions that allow non-trivial reasoning with inconsistent and incomplete probabilistic information. We also formalise reasoning with non-standard probabilities and belief functions in two ways. First, using a calculus of linear inequalities, akin to the one presented in [6]. Second, as a two-layered modal logic wherein reasoning with evidence (the outer layer) utilises paraconsistent expansions of Łukasiewicz logic. The second approach is inspired by [1]. We prove completeness for both kinds of calculi and show their equivalence by establishing faithful translations in both directions.

Keywords: belief functions; Belnap–Dunn logic; two-layered modal logics; paraconsistent logics; Łukasiewicz logic

Motivation and goal. Probabilities have been developed, mostly in the context of classical logic, to model reasoning based on probabilistic information. Belief functions are a generalisation of probabilities for situations where one is not able to give the exact probability of an event, but an approximation in the terms of an upper/lower bound. They were developed based on classical reasoning to handle situations with incomplete information, but they often produce counter-intuitive results when formalising situations involving contradictory information.

In [8] the authors propose a generalisation of probabilities for reasoning based on Belnap Dunn logic BD. In this paper, we extend their work and propose a generalisation of classical belief functions which is based on BD, and

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provide two-layered modal logics extending BD for reasoning about probabilities and belief functions. We focus on finite structures, therefore we consider logics over a finite set of atomic propositions and finite algebras.

Representation of uncertainty

Probabilistic reasoning based on incomplete and inconsistent information. The main idea behind Belnap-Dunn logic is to treat positive and negative information independently. A BD model is a tuple $\mathcal{M} = \langle S, v^+, v^- \rangle$ where S is a finite set of states, $v^+, v^- : S \times \mathrm{Prop} \to \{0,1\}$ are valuations encoding respectively the positive and negative information respectively. A *probabilistic model* $\mathcal{M} = \langle S, \mu, v^+, v^- \rangle$ extends a BD model with a probability measure μ on the powerset algebra $\mathcal{P}S$.

Let us call $|\varphi|^+_{\mathcal{M}} = \{s \in \Sigma : v^+(\varphi) = 1\}$ and $|\varphi|^-_{\mathcal{M}} = \{s \in \Sigma : v^-(\varphi) = 1\}$ the positive and negative extensions of φ respectively. They are mutually definable via negation: $|\varphi|^-_{\mathcal{M}} = |\neg\varphi|^+_{\mathcal{M}}$. The *non-standard probability function* based on $\mathcal{M}$ is defined as $\mathsf{p}^+_\mu(\varphi) := \mu(|\varphi|^+_{\mathcal{M}})$ and represents the positive probabilistic evidence for φ. (Positive) non-standard probabilities satisfy the following three axioms: (i) $0 \leq \mathsf{p}^+(\varphi) \leq 1$, (ii) $\{\mathsf{p}^+(\varphi) \leq \mathsf{p}^+(\psi) \mid \varphi \vdash_{\mathsf{BD}} \psi\}$, and (iii) $\mathsf{p}^+(\varphi \wedge \psi) + \mathsf{p}^+(\varphi \vee \psi) = \mathsf{p}^+(\varphi) + \mathsf{p}^+(\psi)$. We can define negative non-standard probability in a similar manner as $\mathsf{p}^-_\mu(\varphi) = \mu(|\varphi|^-_{\mathcal{M}})$, but from a formal point of view it is sufficient to work with the positive one as $\mathsf{p}^-(\varphi) = \mathsf{p}^+(\neg\varphi)$. Notice that unlike in the classical case, one can no longer prove that $\mathsf{p}^+(\varphi) + \mathsf{p}^+(\neg\varphi) = 1$.

Evidential reasoning via belief functions and Dempster-Shafer combination rule. Here, we generalise the framework introduced in [8] to belief functions. We interpret belief functions on De Morgan algebras and propose a logic to reason with belief function based on BD. Belief functions [9] allow us to reason with the lower approximation of the probability of an event rather than with its exact probability. A *belief function* $\mathsf{bel} : \mathcal{L} \to [0,1]$ on a bounded lattice is a map such that: for every $a, a_1, \ldots a_k, \ldots a_n \in \mathcal{L}$, we have: (1) $\mathsf{bel}(\bot) = 0$ and $\mathsf{bel}(\top) = 1$; (2) for every $a \in \mathcal{L}$, $0 \leq \mathsf{bel}(a) \leq 1$; (3) for every $k \geq 1$, and every $a_1, \ldots, a_k \in \mathcal{L}$,

$$\mathtt{bel}\left(\bigvee_{1 \leq i \leq k} a_i\right) \geq \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \mathtt{bel}\left(\bigwedge_{j \in J} a_j\right). \tag{1}$$

Recall that a *mass function* $\mathsf{m} : \mathcal{L} \to [0,1]$ on a bounded lattice $\mathcal{L}$ is a map such that: $m(\bot) = 0$ and $\sum_{a \in \mathcal{L}} m(a) = 1$. Every mass function $\mathsf{m} : \mathcal{L} \to [0,1]$ defines a belief function bel_m as follows: for every $a \in \mathcal{L}$, $\mathsf{bel}_\mathsf{m}(a) = \sum_{b \leq a} \mathsf{m}(b)$. Equivalently, for every belief function bel, one can compute its associated mass function m_bel such that the previous equation holds.

Conceptually, mass of a encodes the amount of information provided exactly about a, while the belief of a represents the amount of all the evidence supporting a. Dempster-Shafer combination rule [9] provides a method to aggregate belief functions based on their associated mass functions. Let $m_1, m_2 : \mathcal{L} \to [0,1]$ be two mass functions, their aggregation $m_{1\oplus 2}$ is: $\forall a \in \mathcal{L}$,

$$m_{1\oplus 2}(a) = \frac{1}{1-K} \sum_{b \wedge c = a \neq \bot} m_1(b) m_2(c), \tag{2}$$

where $K = \sum_{b\wedge c=\bot} m_1(b)m_2(c)$. K is a normalisation term that encodes the fact that any fully contradictory information between m_1 and m_2 is ignored. For this reason the combination rule can give very counter intuitive results as demonstrated in the following example.

Example: Two disagreeing doctors. A patient has disease a, b or c and one assumes that he has only one of these diseases. A first expert thinks that the patient has disease a (resp. b and c) with probability 0.9 (resp. 0.1 and 0). This opinion is encoded via the mass function $m_1 : \mathcal{P}(\{a,b,c\}) \to [0,1]$ such that $m_1(a) = 0.9$, $m_1(b) = 0.1$ and $m_1(c) = 0$. A second expert thinks that he has disease a (resp. b and c) with probability 0 (resp. 0.1 and 0.9). This opinion is encoded via the mass function $m_2 : \mathcal{P}(\{a,b,c\}) \to [0,1]$ such that $m_2(a) = 0.9$, $m_2(b) = 0.1$ and $m_2(c) = 0$. Using (2), one gets the following aggregated mass function $\mathsf{m}_{1\oplus 2} : \mathcal{P}(\{a,b,c\}) \to [0,1]$: for every $x \in \mathcal{P}(\{a,b,c\})$, we have $\mathsf{m}_{1\oplus 2}(x) = 1$ if $x = b$, 0 otherwise. This means that $\mathsf{bel}_{1\oplus 2}(b) = 1$ and $\mathsf{bel}_{1\oplus 2}(a) = \mathsf{bel}_{1\oplus 2}(c) = 0$. Therefore while both experts agreed that b was unlikely and that it is highly likely that the patient has an other disease (a or c), one concludes that the patient must have disease b. This results follows from the fact that a, b and c are considered mutually incompatible. Notice that the term K that measure 'contradiction' is equal to 0.99 which means that most of the information given by the experts was ignored.

The same computation over the De Morgan algebra $\mathcal{D}$ generated by $\{a,b,c\}$ leads to a very different conclusion. If one considers the mass functions $m_1 : \mathcal{D} \to [0,1]$ such that $m_1(a \wedge \neg b \wedge \neg c) = 0.9$, $m_1(\neg a \wedge b \wedge \neg c) = 0.1$ and $m_1(\neg a \wedge \neg b \wedge c) = 0$ and $m_2 : \mathcal{D} \to [0,1]$ such that $m_1(a \wedge \neg b \wedge \neg c) = 0$, $m_1(\neg a \wedge b \wedge \neg c) = 0.1$ and $m_1(\neg a \wedge \neg b \wedge c) = 0.9$, one gets the following aggregated mass function $\mathsf{m}_{1\oplus 2}$ (we represent only the elements in $\mathcal{D}$ with non-zero mass):

	$\mathsf{m}_{1\oplus 2}$
$\neg a \wedge b \wedge \neg c$	0.01
$a \wedge \neg a \wedge b \wedge \neg b \wedge \neg c$	0.09
$a \wedge \neg a \wedge \neg b \wedge c \wedge \neg c$	0.81
$\neg a \wedge b \wedge \neg b \wedge c \wedge \neg c$	0.09

Therefore, one reaches the conclusion that one has strong contradictory information regarding a and c and that b is most probably not the case, since $\mathsf{m}_{1\oplus 2}(a \wedge \neg a \wedge \neg b \wedge c \wedge \neg c) = 0.81$. This tells us to search for additional information to figure out whether the patient has disease a or c. This observation leads us to think that in presence of highly conflicting information, it is more relevant to interpret belief functions over De Morgan algebras and therefore to reason with BD rather than with classical logic.

Two-layered Belnapian Logics for probabilities and belief functions

Two-layer logics for reasoning under uncertainty were introduced in [6,7], and developed further within an abstract algebraic framework by [5] and [2]. Two-layer logics separate two layers of reasoning: the inner layer consists of a logic chosen to reason about events (often classical propositional logic interpreted over sets of possible worlds), the connecting modalities are interpreted by a chosen uncertainty measure on propositions of the inner layer (typically a probability or a belief function), and the outer layer consists of a logical framework to reason about probabilities or beliefs. The modalities apply to inner layer formulas only, to produce outer layer atomic formulas, and they never nest. Logics introduced in [6] use classical propositional logic on the lower layer, and reasoning with linear inequalities on the upper layer. [7] on the other hand uses Łukasiewicz logic on the outer layer, to capture the quantitative, many-valued reasoning about probabilities within a propositional logical language. Building on that idea, and having in mind the two-dimensionality of uncertain information (e.g. positive and negative probabilities), we have introduced a two layer modal logic to reason with non-standard probabilities in [4]. There a two-dimensional extension of Łukasiewicz logic containing an additional De Morgan negation has been proposed. Another two-dimensional extension of Łukasiewicz logic, where De Morgan negation of implication behaves differently, has been introduced in [3], and both logics (which we denote $\text{Ł}^2(\rightarrow)$ and $\text{Ł}^2(\multimap)$) were shown to be coNP complete using constraint tableaux calculi. We provide Hilbert-style axiomatizations for both the logics, which are finitely standard strong complete w.r.t. the twist product of the standard MV algebra $[0,1]_{\text{Ł}}^{\bowtie}$.

In this talk, we consider two-layered logics which use BD as the inner layer, a single unary probability modality P (or a belief modality B) applied to BD formulas, and $\text{Ł}^2(\rightarrow)$ or $\text{Ł}^2(\multimap)$ on the outer layer. The inner formulas are interpreted over a BD model $\mathcal{M} = \langle S, v^+, v^- \rangle$, the atomic modal formulas are interpreted in $[0,1]_{\text{Ł}}^{\bowtie}$ via a given probability (or belief) function on $\mathcal{P}S$ as

$$v^{\mathcal{M}}(P\varphi) = (\mathsf{p}(|\varphi|^+_{\mathcal{M}}), \mathsf{p}(|\varphi|^-_{\mathcal{M}})) \qquad v^{\mathcal{M}}(B\varphi) = (\mathsf{bel}(|\varphi|^+_{\mathcal{M}}), \mathsf{bel}(|\varphi|^-_{\mathcal{M}})),$$

and outer formulas are interpreted in the algebra $[0,1]_{\text{Ł}}^{\bowtie}$ following the semantics of the chosen variant of Ł^2.

We present the resulting two-layer logics via Hilbert-style two-layer axiomatizations of the form $\langle \mathsf{BD}, M_p, \text{Ł}^2 \rangle$, and $\langle \mathsf{BD}, M_b, \text{Ł}^2 \rangle$, and prove their completeness. Here, BD is an axiomatization of the logic BD, and M_p, M_b are sets of modal axioms and rules capturing the behaviour of the P or B modality respectively. Axioms M_p of probability for example look as follows:

$$\vdash_{\text{Ł}^2} P\neg\varphi \leftrightarrow \neg P\varphi \quad \{\vdash_{\text{Ł}^2} P\varphi \rightarrow P\psi \mid \varphi \vdash_{\mathsf{BD}} \psi\}$$
$$\vdash_{\text{Ł}^2} P(\varphi \vee \psi) \leftrightarrow (P\varphi \ominus P(\varphi \wedge \psi)) \oplus P\psi,$$

where $\oplus, \ominus$ are connectives definable in Ł^2 as in Lukasiewicz logic, corresponding (point-wise) to truncated addition/subtraction on $[0,1]$ respectively.

In the case we deal with belief functions, the first two axiom schemes for B modality stay in place. While expressing the probability axioms in Łukasiewicz logic as above is rather straightforward (see [7,4]), formulating the belief k-monotonicity axioms is less so. We define a sequence of outer formulas γ_n in propositional letters of the inner language $p_1, \ldots, p_n$ inductively as follows:

$$\gamma_1 \coloneqq Bp_1 \quad \gamma_{n+1} \coloneqq \gamma_n \oplus (Bp_{n+1} \ominus \gamma_n[B\psi : B(\psi \wedge p_{n+1}) \mid B\psi \text{ atoms of } \gamma_n]),$$

where $\gamma_n[B\psi : B(\psi \wedge p_{n+1}) \mid B\psi \text{ modal atoms of } \gamma_n])$ is the result of replacing each modal atom $B\psi$ in γ_n with the modal atom $B(\psi \wedge p_{n+1})$ (semantically, it is a relativisation of the corresponding belief function to the sets $|p_{n+1}|^{+-}$). The n-th belief function axiom (i.e., the n-monotonicity) is expressed by substitution instances (substituting inner formulas for the atomic letters $p_1, \ldots, p_n$) of

$$\alpha_n \coloneqq \gamma_n \to B(\bigvee_{i=1}^{n} p_n).$$

Additionally to Ł^2-based logics, we present a two-layer logic for belief functions based on BD on the lower level, and two-dimensional reasoning about linear inequalities on the upper level. We will relate the two formalism by way of translation, following [1], and we will compare the resulting logic to the one introduced in [10].

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A Sahlqvist-style Correspondence Theorem for Linear-time Temporal Logic

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1 Introduction

One of the most well-known results in the model theory of modal logic is that modal languages are rich enough to express (first-order) conditions on Kripke frames. Results along this line have been known as *Correspondence Theory* [12,3]. Since the 1970's, much research in modal logic has been devoted to identifying classes of formulas for which such first-order correspondents exist, including algorithms for their automatic computation. The classic result by H. Sahlqvist [10] identifies a significant class of modal formulas for which first-order conditions – or Sahlqvist correspondents – can be found in an effective, algorithmic way. Since then, correspondence theory has been successfully extended to more complex and expressive modal languages [7,11,13].

Contribution. In this paper we develop a Sahlqvist-style correspondence theorem for Linear-time Temporal Logic (LTL), which is nowadays one of the most widely-used formal languages for temporal specification [1]. To accommodate the enhanced expressiveness, we extend the class of Sahlqvist formulas with some additional conditions. Our main result is to prove the correspondence of such Sahlqvist formulas in LTL to frame conditions that are definable in a first-order language.

Related Work. As we mentioned above, Sahlqvist correspondence theorem has been extended in a number of different directions, mainly considering more expressive modal languages. In [7] a correspondence theorem is proved for temporal modal logic, whereas in [13,2] similar results are proved for the μ-calculus and modal fixed-point logic respectively. More recently, correspondence results have been proved for hybrid logics [6], distributive modal logics [8], and polyadic

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modal logics [9]. Some efforts have also been applied to the problem of finding more general and efficient algorithms to compute first-order correspondents of modal formulas [4,5,14], including [11] mentioned earlier. Still, to the best of our knowledge, no comparable result has been proved for the kind of temporal logics used in the specification and verification of reactive and distributed systems [1]. We deem such a result of interest to theoreticians and practitioners in modal logics alike.

Due to restrictions on space, we omit all proofs of stated results. An extended version of the paper with full proofs can be found here: https://www.dropbox.com/s/r1mrvi1tpr57mz7/paper.pdf?dl=0.

2 Preliminaries: Linear-time Temporal Logic

Due to lack of space, we refer to [1] for full details on the syntax and semantics of LTL. We consider also a variant of LTL, that we call LTL'. Its syntax is defined in BNF as follows:

$$\phi = Prop \mid \bot \mid \top \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid G\phi \mid F_x\phi \mid \widehat{G}_{w,w'}\phi \mid X\phi$$

where w and w' ($w \neq w'$) are states in the set W of possible worlds (which serves as the model of LTL and LTL'), and x is a variable over states.

To provide a semantics to LTL, we consider *transition systems* $T = (S, \rightarrow)$, where S is a set of *states*, and the *transition relation* $\rightarrow \subseteq S \times S$ is a binary relation on S, typically assumed to be *serial*. Then, a *path* is an infinite sequence $s_1, s_2, s_3, \ldots$, where for all $i \in \mathbb{N}$, $s_i \rightarrow s_{i+1}$.

We now define models for LTL. Let W be the set of all paths in T; and $\leqslant$, $<$, and $\mathbf{S}$ are binary relations on W, introduced as follows. Let $w = s_1, s_2, s_3, \ldots$ and $v = s'_1, s'_2, s'_3, \ldots$ be paths in T, then $w \leqslant v$ iff for some $i \geqslant 1$, $s_i = s'_1$ and for all $j > 0$, $s_{i+j} = s'_{1+j}$. Then, $w < v$ iff $w \leqslant v$ and $w \neq v$. Further, $\mathbf{S}$ means *successor*: $v = \mathbf{S}(w)$ iff for all $i > 0$, $s'_i = s_{i+1}$. When the context is clear, we sometimes simply write $R(w, v)$ for $w < v$, $w \leqslant v$ or $v = \mathbf{S}(w)$.

A *model* for LTL is a tuple $M = (T, h)$, where T is a transition system, and $h : Prop \rightarrow 2^S$ is an *assignment function* from atoms to set of states in S. We lift the assignment h from states to paths so that $w \in h(q)$ iff $s_1 \in h(q)$.

Definition 2.1 [Satisfaction] Given a model M, path w, and formula ϕ in LTL', the *satisfaction relation* $\models$ is defined as follows: the clauses for $\neg, \wedge, \vee, G, X$ are standard, whereas

$(M, w) \models q$	iff	$w \in h(q)$
$(M, w) \models F_x\phi$	iff	for some $x \in W$, $w \leqslant x$ and $(M, x) \models \phi$
$(M, w) \models \widehat{G}_{w,w'}\phi$	iff	for all $u \in W, w \leqslant u < w'$ implies $(M, u) \models \phi$

Hereafter we use $w \models \phi$ as an abbreviation for $(M, w) \models \phi$. We write $[\phi]^h_w = 1$ iff $(M, w) \models \phi$ for $M = (T, h)$.

We now show that there exists a truth preserving translation τ from LTL to LTL'.

Definition 2.2 [Translation] Let τ be the translation from LTL to LTL' defined as follows: for $\circ \in \{\wedge, \vee\}$, $\tau(\phi_1 \circ \phi_2) = \tau(\phi_1) \circ \tau(\phi_2)$; for $\bullet \in \{\neg, F, X\}$,

$\tau(\bullet\phi) = \bullet\tau(\phi)$, and

$$\begin{array}{ll} q & \mapsto q \\ F\phi & \mapsto F_x\tau(\phi) \\ \phi_1 U \phi_2 & \mapsto F_x(\tau(\phi_2) \wedge \widehat{G}_{w,x}\tau(\phi_1)) \end{array}$$

where x is a path variable, and w is the path at which we aim to evaluate the formula.

Lemma 2.3 *Let τ be the translation from LTL to LTL' in Def. 2.2. Then an LTL formula and its translation w.r.t. τ are semantically equivalent.*

Further, we introduce the *standard translation* of formulas in LTL', which mirrors their semantics. For every atom $q \in Prop$, we introduce a predicate symbol Q. For an arbitrary formula ϕ in LTL', we denote the first-order standard translation of ϕ at w as $ST_w(\phi)$.

Definition 2.4 [Standard Translation] The standard translation $ST_w(\phi)$ of formula ϕ at path w is inductively defined as case of ϕ: for propositional connectives, we refer to [3]; as for the rest, we have

$$\begin{array}{lcl} q & : & Q(w) \\ G\phi & : & \forall v(w \leqslant v \rightarrow ST_v(\phi)) \\ F_x\phi & : & \exists x(w \leqslant x \wedge ST_x(\phi)) \\ \widehat{G}_{s,s'}\phi & : & \forall v(s \leqslant v < s' \rightarrow ST_v(\phi)) \\ X\phi & : & ST_{\mathbf{S}(w)}(\phi) \end{array}$$

Let T be a transition system and $w \in W$. An LTL' formula $\phi(q_1, q_2, \ldots, q_k)$ is said to *correspond* to a formula φ in second-order logic at w whenever ϕ are φ are both true at w in T.

Lemma 2.5 *An LTL' formula $\phi(q_1, \ldots, q_k)$ corresponds to $\forall Q_1 \ldots \forall Q_k ST_w(\phi)$, where $ST_w(\phi)$ is the (first-order) standard translation of $\phi[w]$.*

In light of this lemma, we will be using semantics and standard translation interchangeably in this paper.

3 Sahlqvist Formulas for LTL

In this section, we introduce two particular types of formulas that play key roles in the construction of Sahlqvist formulas: boxed formulas and negative formulas. We prove their monotonicity in Lemma 3.3 and introduce Sahlqvist formulas for LTL in Def. 3.4.

Definition 3.1 [Accessibility Relation R^n] We define boxed formulas $\boxplus^n q$ and the accessibility relation R^n by induction on $n \in \mathbb{N}$.

Base case: if $n = 0$, then $\boxplus^n q = q$ is a boxed formula and $R^0(w, v)$ iff $w = v$.

Inductive cases: assume $\boxplus^n q$ is a boxed formula, then

- $\boxplus^{n+1} q = G \boxplus^n q$ is a boxed formula, and $R^{n+1}(w, v)$ iff for some $u \in W$, $w \leqslant u$ and $R^n(u, v)$.
- $\boxplus^{n+1} q = X \boxplus^n q$ is a boxed formula, and $R^{n+1}(w, v)$ iff $R^n(\mathbf{S}(w), v)$.

• $\boxplus^{n+1}q = \widehat{G}_{s,s'} \boxplus^n q$ is a boxed formula, and $R^{n+1}(w,v)$ iff for some $u \in W$, $s \leqslant u < s'$ and $R^n(u,v)$.

By Def. 3.1 we can prove the following auxiliary result concerning boxed formulas.

Lemma 3.2 (Boxed Formulas Lemma) *Let $\boxplus^n q$ be an LTL' boxed formula with n boxed operators appearing in front of atom q (with n possibly equal to 0). Then $w \vDash \boxplus^n q$ iff for all $v \in W$, $R^n(w,v)$ implies $v \vDash q$.*

This lemma shows that the standard translation of every boxed formula $\boxplus^n q$ can be written in the form of $\forall v, R(w,v) \to Q(v)$ using a unique relation R. This construction will be invaluable in defining the minimal assignment for Sahlqvist formulas.

Similarly to standard modal logic, LTL' *positive formulas* ϕ can be defined as the ones constructed from atoms using $\wedge$, $\vee$, G, F_x, $\widehat{G}_{w,w'}$, X only:

$$\phi = Prop \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid G\phi \mid F_x\phi \mid \widehat{G}_{w,w'}\phi \mid X\phi$$

An LTL' *negative formula* has one of the two following forms:

(i) $\neg\phi$, where ϕ is an LTL' positive formula;

(ii) $\widehat{G}_{w,w'}N$, where N is an LTL' negative formula.

Lemma 3.3 (Monotonicity) *Let ϕ be an LTL' positive formula, $q_1, \ldots, q_k$ be the atoms appearing in ϕ, and h_1 and h_2 be assignments. If for all q_j, $h_1(q_j) \subseteq h_2(q_j)$, then $h_1(\phi) \subseteq h_2(\phi)$.*

By Lemma 3.3 we obtain that, whenever N is an arbitrary LTL' negative formula, $q_1, \ldots, q_k$ are the atomic variables appearing in P, and h_1, h_2 are two random assignments, if for all q_j, $h_1(q_j) \subseteq h_2(q_j)$, then $h_2(N) \subseteq h_1(N)$.

We finally provide the definitions of Sahlqvist formula for LTL and LTL'.

Definition 3.4 [LTL Sahlqvist Formulas] Suppose β is an LTL boxed formula or negative formula. Then, we define LTL *untied formula* ϕ as follows:

$\phi = A_{LTL} \mid N_{LTL} \mid \beta U \phi \mid \phi \wedge \phi$

The LTL *Sahlqvist formulas* are the conjunction of negations of LTL untied formulas.

Definition 3.5 [LTL' Sahlqvist Formulas] An LTL' untied formula is constructed from LTL' boxed formulas and LTL' negative formulas using only F_x and conjunction:

$\phi = A_{LTL'} \mid N_{LTL'} \mid \phi \wedge \phi \mid F_x\phi$

As before, LTL' Sahlqvist formulas are the conjunctions of negations of LTL' untied formulas.

4 Correspondence Theorem

In this section we present the proof of the correspondence theorem for LTL. By embedding LTL Sahlqvist formulas into LTL' Sahlqvist formulas, we only need to show that the theorem holds for the latter. We start by showing that the translation τ from LTL to LTL' preserves Sahlqvist formulas. Then we introduce the main lemma crucial to the theorem.

Lemma 4.1 *Let τ be the translation from LTL to LTL' in Def. 2.2. Then the following claims are true:*

(1) *The translation of an LTL untied formula w.r.t. τ is an LTL' untied formula.*

(2) *An LTL untied formula and its translation w.r.t. τ are semantically equivalent.*

Whenever two formulas are semantically equivalent, they have the same frame conditions. Therefore, having shown that for each LTL Sahlqvist formula, a semantically equivalent LTL' formula exists and is also Sahlqvist, we can conclude the following lemma:

Lemma 4.2 *If every LTL' Sahlqvist formula locally corresponds to a first order formula, then every LTL Sahlqvist formula locally corresponds to a first order formula.*

The remaining propositions prove the correspondence of LTL Sahlqvist formulas to first-order frame conditions.

Lemma 4.3 (Main Lemma) *Let E be an LTL' untied formula, w is a state, and h_0 is the minimal assignment of ϕ at w (possibly empty). Let h be an assignment. If there exists an assignment g and a state w such that $[E]^g_w = 1$, then the following are equivalent:*

(a) *For all $q_j \in \{q_1, \ldots, q_k\}$, $h_0(q_j) \subseteq h(q_j)$.*

(b) $[B]^h_w = 1$.

where B is defined as follows: if there are only boxed formulas in E, then $B = E$; if there are any negative formulas in E, then B is obtained from E by replacing all occurrences of negative formulas $N_1, N_2, \ldots, N_m$ in E by $\top$. Also, h_0 is the minimal assignment, defined in a similar way as in [7].

Theorem 4.4 (LTL Correspondence Theorem) *Let S be an LTL' Sahlqvist formula, then the local correspondent of $S[w]$ can be expressed in first-order terms, i.e., $\forall Q_1, \ldots, \forall Q_k, ST_w(S(q_1, \ldots, q_k))$ has a first-order correspondent.*

5 Conclusions

In this paper we introduced a notion of Sahlqvist formula for the Linear-time Temporal Logic LTL and proved a Sahlqvist correspondence theorem for this language. In fact, LTL' Sahlqvist formulas are very similar to the Sahlqvist formulas of standard modal logic to the extent that the proof for the completeness property [3,7] for Sahlqvist formulas almost identically applies to the LTL' Sahlqvist formulas.

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Hyperintensional models for non-congruential modal systems

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Abstract

Hyperintensional models for modal logic consitute a flexible semantic framework which allows for a rigorous distinction between the proposition expressed by a formula (its intension) and the semantic content of a formula (its hyperintension). In this work we show that any modal system extending the Propositional Calculus is semantically characterized by a class of hyperintensional models; in particular, we provide a detailed discussion of soundness and completeness results for some examples of non-congruential modal systems.

Keywords: Hyperintensional models, non-congruential systems, modal logic.

1 Introduction

This work concerns applications of a semantics for modal logic introduced in [9] and based on structures called *hyperintensional models*. In the first part of the presentation we discuss some philosophical ideas behind the approach; in particular, the difference between the proposition expressed by a formula (its *intension*) and the semantic content of a formula (its *hyperintension*), which is captured in a precise way in hyperintesional models. Next, we illustrate the technical core of the framework and provide a general characterization result for arbitrary modal extensions of the Propositional Calculus (PC). Moreover, we analyse examples of non-congruential modal systems and provide a simplified characterization for them in terms of a subclass of our structures called Boolean-content models. In the light of the results obtained, we conclude the presentation by claiming that hyperintensional models constitute a *basic*, *general* and *unifying* framework for the interpretation of modal logic.

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The use of non-congruential systems of modal logic is crucial to represent contexts of reasoning that can be called “logically hyperintensional” [3]. These are contexts in which two formulas that are logically equivalent cannot be always substituted *salva veritate* in the scope of a modal operator. Epistemic modals are well-known examples of hyperintensional contexts; for instance, the set of formulas representing what is explicitly known by a subject endowed with bounded rationality is not closed under logical equivalence.

Hyperintensional models are a recent framework representing an alternative to various semantics for non-congruential systems formulated over the years. Some of these semantics are tailored to specific classes of systems (see, e.g., [2], [1] and [6]); others aim at constituting a general framework (see, e.g., [7], [8] and [4]). Hyperintensional models fall in the latter category.

2 Formal framework

Let a *propositional language* $\mathcal{P}$ contain a countable set of propositional variables Pr and the set of connectives $Con_{\mathcal{P}} = \{\wedge, \vee, \rightarrow, \bar{0}\}$, where $\wedge, \vee, \rightarrow$ are binary and $\bar{0}$ is zero-ary. The language $\mathsf{Mod}(\mathcal{P})$, which is a modal extension of $\mathcal{P}$, contains Pr and $Con_{\mathsf{Mod}(\mathcal{P})} = Con_{\mathcal{P}} \cup \{\Box\}$, where $\Box$ is unary. The set of formulas of $\mathcal{X}$, for $\mathcal{X} \in \{\mathcal{P}, \mathsf{Mod}(\mathcal{P})\}$, is denoted as $Fm_{\mathcal{X}}$. We define $\neg\varphi := \varphi \rightarrow \bar{0}$ and $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

An *$\mathcal{X}$-type algebra* is any algebra $\mathbf{A} = (A, \{c^{\mathbf{A}} \mid c \in Con_{\mathcal{X}}\})$. $Fm_{\mathcal{X}}$ can be seen as an $\mathcal{X}$-type algebra. Given two algebras $\mathbf{A}$ and $\mathbf{B}$, an *$\mathcal{X}$-homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a mapping that commutes with all elements of $Con_{\mathcal{X}}$.

Definition 2.1 A *hyperintensional model* is a tuple $\mathfrak{M} = (W, \mathbf{C}, O, N, I)$ s.t.:

- W is a non-empty set;
- $\mathbf{C}$ is a $\mathcal{P}$-type algebra;
- O is a $\mathcal{P}$-homomorphism from $Fm_{\mathsf{Mod}(\mathcal{P})}$ to $\mathbf{C}$;
- N is a function from W to subsets of (the universe of) $\mathbf{C}$;
- I is a $\mathcal{P}$-homomorphism from $\mathbf{C}$ to the power-set algebra over W such that, for all $\varphi \in Fm_{\mathsf{Mod}(\mathcal{P})}$ and $w \in W$

$$w \in I(O(\Box\varphi)) \iff O(\varphi) \in N(w) \tag{1}$$

We define E as the composition of O and I. A formula φ is *valid in* $\mathfrak{M}$ iff $E(\varphi) = W$. We will sometimes write $(\mathfrak{M}, w) \models \varphi$ instead of $w \in E(\varphi)$, where E is defined over $\mathfrak{M}$. Informally, $\mathbf{C}$ is a set of “semantic contents” of declarative sentences endowed with some algebraic structure. $O(\varphi)$ is the semantic content assigned to φ. The idea at the basis of these models is that the semantic value of an expression results from the two step procedure illustrated in Fig. 1.

3 Fundamental characterization result

A logic is here regarded either as an axiomatic system or as a set of theorems that represents an extension of PC closed under Modus Ponens and Uniform Substitution. Given a logic L based on a language $\mathcal{X}$, φ^L is the set of all

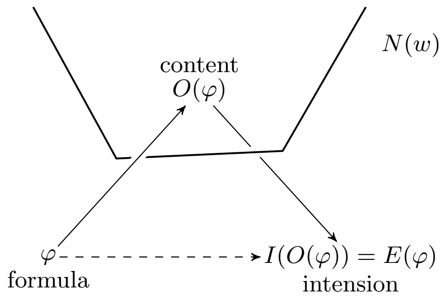

Fig. 1. Computing the semantic value of an expression.

maximal consistent L-theories Γ s.t. $\varphi \in \Gamma$. Moreover, $Fm^L_{\mathcal{X}} = \{\varphi^L \mid \varphi \in Fm_{\mathcal{X}}\}$.

Theorem 3.1 (General Characterization) *For each logic L over $\mathsf{Mod}(\mathcal{P})$, there is a hyperintensional model $\mathfrak{M}^L$ s.t., for all ϕ, $\phi \in L$ iff ϕ is valid in $\mathfrak{M}^L$.*

Proof. An adaptation of the usual canonical model construction, where the model $\mathfrak{M}^L = (W^L, \mathbf{C}^L, O^L, N^L, I^L)$ is such that: W^L is the set of all maximal consistent L-theories, $\mathbf{C}^L = Fm_{\mathsf{Mod}(\mathcal{P})}$, $N^L(\Gamma) = \{\varphi \mid \Box\varphi \in \Gamma\}$, $O^L(\varphi) = \varphi$ and $I^L(\varphi) = \varphi^L$.

$\Box$

4 Examples of non-congruential systems

Properties of hyperintensional models can be added in a modular way to obtain a semantic characterization for specific non-congruential modal systems. Some examples are illustrated below.

Definition 4.1 A *Boolean-content model* is a hyperintensional model where $\mathbf{C}$ is a Boolean algebra.

Let $x, y \in U$ (where U is the carrier of the Boolean algebra $\mathbf{C}$). We define $x \leq^{\mathbf{C}} y$ as $x \vee^{\mathbf{C}} y = y$. A Boolean-content model is said to be:

- *monotonic* iff, for all $w \in W$, $x \leq^{\mathbf{C}} y$ only if $x \in N(w) \implies y \in N(w)$;
- *regular* iff it is monotonic and, for all $w \in W$, $x, y \in N(w) \implies x \wedge^{\mathbf{C}} y \in N(w)$;
- *normal* iff it is regular and, for all $w \in W$, $O(\overline{1}) \in N(w)$;
- *N-consistent* iff, for all $w \in W$, $x \in N(w) \implies \neg^{\mathbf{C}} x \notin N(w)$;
- *N-factive* iff for all $w \in W$, $x \in N(w) \implies w \in I(x)$.

We analyse the following non-congruential systems over $\mathsf{Mod}(\mathcal{P})$, all of which contain the Propositional Calculus (PC) and are closed under Modus Ponens:

- $B0$ is the weakest system that is closed under the rule (RE$_{PC}$) $\dfrac{\varphi \leftrightarrow \psi \in PC}{\Box\varphi \leftrightarrow \Box\psi}$;

- $B1$ is the weakest system that is closed under the rule (RM$_{PC}$) $\dfrac{\varphi \to \psi \in PC}{\Box\varphi \to \Box\psi}$;
- $C1$ is the weakest system including the axiom (K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ that is closed under (RM$_{PC}$);
- $D1$ is the weakest system including the axioms (K) and (D) $\Box\varphi \to \neg\Box\neg\varphi$ that is closed under (RM$_{PC}$).
- $E1$ is the weakest system including the axioms (K) and (T) $\Box\varphi \to \varphi$ that is closed under (RM$_{PC}$).
- $S0.5^o$ is the weakest system including the axiom (K) that is closed under the rule (RN$_{PC}$) $\dfrac{\varphi \in PC}{\Box\varphi}$;
- $S0.5$ is the weakest system including the axioms (K) and (T) that is closed under the rule (RN$_{PC}$).

Any logic L in this list is closed under inference rules whose formulation makes explicit reference to a different logic, i.e. PC, which is properly contained in L. The role played by PC will be taken into account in the algebra of contents of the canonical models introduced below for these logics. We stress that, in accordance with the previously introduced notation, φ^{PC} denotes the set of all maximal consistent PC-theories containing φ.

Definition 4.2 Let L be a logic over $\mathsf{Mod}(\mathcal{P})$. The *$PC$-content canonical model for L* is a tuple $\mathfrak{M}^{PC/L} = (W^{PC/L}, \mathbf{C}^{PC/L}, O^{PC/L}, N^{PC/L}, I^{PC/L})$ s.t.:

- $W^{PC/L}$ is the set of all maximal consistent L-theories;
- $\mathbf{C}^{PC/L} = (Fm^{PC}_{\mathsf{Mod}(\mathcal{P})}, \{c^{\mathbf{C}^{PC/L}} \mid c \in Con_{\mathcal{P}}\})$; where $c^{\mathbf{C}^{PC/L}}(\varphi_1^{PC}, \ldots, \varphi_n^{PC}) = \big(c(\varphi_1, \ldots, \varphi_n)\big)^{PC}$;
- $O^{PC/L}(\varphi) = \varphi^{PC}$;
- $N^{PC/L}(\Gamma) = \{O^{PC/L}(\varphi) \mid \Box\varphi \in \Gamma\} = \{\varphi^{PC} \mid \Box\varphi \in \Gamma\}$;
- $I^{PC/L}(O^{PC/L}(\phi)) = I^{PC/L}(\varphi^{PC}) = \varphi^L$.

Lemma 4.3 *Let L be a logic over $\mathsf{Mod}(\mathcal{P})$; then, $\mathfrak{M}^{PC/L}$ is a Boolean-content model.*

Proof. $O^{PC/L}$ is a $\mathcal{P}$-homomorphism by definition of c^{PC}. $\mathbf{C}^{PC/L}$ is a Boolean algebra due to the definition of PC (in fact, a set algebra). Next, $I^{PC/L}$ is a $\mathcal{P}$-homomorphism due to the properties of maximal consistent L-theories. $\Box$

Lemma 4.4 *For each logic L over $\mathsf{Mod}(\mathcal{P})$, each $\Gamma \in W^{PC/L}$ and each φ:*

$$\varphi \in \Gamma \iff (\mathfrak{M}^{PC/L}, \Gamma) \models \varphi \tag{2}$$

Proof. $E^{PC/L}(\varphi) = I^{PC/L}(O^{PC/L}(\varphi)) = I^{PC/L}(\varphi^{PC}) = \varphi^L$. $\Box$

Theorem 4.5 (Modular Characterization)

(i) *$\varphi \in B0$ iff φ is valid in all Boolean-content models.*

(ii) $\varphi \in B1$ *iff* φ *is valid in all monotonic Boolean-content models.*

(iii) $\varphi \in C1$ *iff* φ *is valid in all regular Boolean-content models.*

(iv) $\varphi \in D1$ *iff* φ *is valid in all regular N-consistent Boolean-content models.*

(v) $\varphi \in S0.5$ *iff* φ *is valid in all normal Boolean-content models.*

(vi) $\varphi \in S0.5^o$ *iff* φ *is valid in all normal and N-factive Boolean-content models.*

5 Final remarks

The results proven have important consequences with respect to applications of the framework, which can be described adopting the terminology in [11]. First, the approach at issue is *basic* since, in its more general form, can be used to semantically characterize the Propositional Calculus formulated over $\mathsf{Mod}(\mathcal{P})$. Second, the approach is *general*, since properties can be added to classes of models in a modular way, thus characterizing modal systems with a different deductive power. Moreover, as it is shown in [9], within this framework one can simulate related approaches developed in the literature; most importantly, hyperintensional models embed Rantala models [7,8], which, in turn, embed models of many other frameworks available [11,10]. For these reasons, hyperintensional models constitute also a *unifying* framework for the analysis of non-congruential modal logics.

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Neighbourhood semantics and axioms for strategic fragment of classical stit logic

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Abstract

Stit (sees to it that) logic is a prominent modal logic of agency used in both philosophy of action and the study of multiagent systems. In stit logic, the modalities describe the properties that are forced by an agent's actions. In comparison with other multi-agent modal logics, the main advantage of stit theories is expressive power. Stit logic allows to study not only statements about agents' abilities to perform certain actions (as it is in variations of Coalition Logic or Propositional Dynamic Logic), but about what choices they make and what they de-facto achieve as well.

Nevertheless, in some occasions such expressivity may be redundant. This paper surveys a specific fragment of classical stit logic, which has only strategic modal operator $[i]\phi$, what stands for the fact that agent i has an ability to see to it that ϕ holds. The neighbourhood semantics for the fragment is presented, accompanied with the soundness, canonicity hence strong completeness results. Furthermore, the paper presents basic considerations on epistemic extension of the presented fragment.

Keywords: Stit logic, neighbourhood semantics, multi-agent systems, logic of agency.

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1 Introduction

Stit (sees to it that) logic is a prominent modal logic of agency used in both philosophy of action and the study of multiagent systems. In stit logic, the modalities describe the properties that are forced by an agent's actions. Actions of individuals are studied via sentences of the form "agent i sees to it that ϕ holds", where the syntactic construction "agent i sees to it that" is treated as a modality $[i : stit]$ (we will use a simpler notation: $[stit]_i$). Stit grows out from a tradition of modal theories of agency [2] and originated from a series of seminal articles by Belnap, Perloff and Xu [2].

In classical stit theory, agent's power to see to it that some state of affairs holds is called a causal ability. The notion ignores any mental attitudes, such as desirability, awareness or purposefulness of the potential action. Casual ability is formalized via a bimodal formula $\Diamond[stit]_i\phi$, i.e. it is possible that agent i sees to it that ϕ obtains. Besides the causal abilities, stit theorists study *epistemic* abilities, i.e. agent's power to knowingly see to it that ϕ. The discussion on epistemic abilities and their logical form is still open: Horty and Pacuit propose to treat epistemic abilities via action types and to introduce a new modal operator $[kstit]_i\phi$, what reads as "agent i executes an action type that she knows to guarantee the truth of ϕ":

> If the epistemic sense of ability requires that some single action must be known by i to guarantee the truth of ϕ, then this must be the action type, not one of its various tokens [6].

While Broersen argue that epistemic ability may be presented simply by adding knowledge operator $K_i\phi$, which satisfies its standard definition from modal epistemic logic. In that case, epistemic ability may be represented as $\Diamond K_i[stit]_i\phi$ – it is possible that agent i knows that she sees to it that ϕ [3].

In the paper, we study causal ability as a single monotonic modal operator. In order to present its axiomatisation, we translate standard stit semantics to neighbourhood one and prove correctness and strong completeness w.r.t. a specific class of neighbourhood frames. We show that the mentioned class is modally invariant with BT+AC frames for strandard stit logic, as it defined in [6].

2 Translating stit semantics to neighbourhood semantics: strategic case

In this section the strategic fragment of $\mathbf{L}_{cstit}$ is surveyed. The fragment is obtained by abandonment of $[stit]_i\phi$ formulas, allowing only statements about agents' abilities. Such statements are treated by a strategic stit modality: $[i]\phi$ stands here for agent i has an ability to achieve ϕ. Strategic stit modality $[i]$ may be seen as an abbreviation for $\Diamond[stit]_i$. In order to present strategic stit fragment and its axiomatisation, the Kripke semantics version of BT+AC frames for classical stit logic as it is presented in [5] should be changed to neighbourhood semantics, since $[i]$ operator loses normality [3].

[2] Detailed overview of the stit prehistory could be found in [9]

[3] The seminal papers showing why normal modal logic is inadequate for ability and how classical modal logic may overcome it: [7], [4]

2.1 Neighbourhood stit: strategic abilities in the one-shot case

The one-shot strategic stit logic's language $\mathbf{L}_{osstit}$ is defined as follows:

$$\phi := p \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi \mid [i]\phi \mid [\exists_i]\phi$$

where $p \in Var, i \in Ags$. $[i]\phi$ stands for "agent i is able to see to it that ϕ" (what intuitively corresponds to $\Diamond[stit]_i\phi$ in classical stit logic), $[\exists_i]\phi$ – "every potential action of agent i guarantees ϕ", what may be seen as $\Box[stit]_i\phi$ equivalent. From the semantic point of view, $[\exists_i]\phi$ is synonymous to $\Box\phi$, just like $\Box[stit]_i\phi$ is equivalent to $\Box\phi$ in classical stit; nevertheless, it holds only if certain condition on frames is satisfied. Generally speaking, the fact that some state of affairs is historically necessary is not equivalent to the fact that some agent is unable not to force that state of affairs: it presupposes that every possible state should be a potential outcome of the agent's actions.

Historical necessity modality, as well as Boolean connectives, have their standard meanings. Dual of every modal operator is defined standardly as well.

It is possible to consider a one-shot model, i.e. a model for a set of agents simultaneously taking some actions at the unique moment.

Definition 2.1 One-shot strategic stit (osstit) frames

$$\mathscr{F} = \langle W, \{Choice_i\}_{i \in Ags} \rangle$$

- $W = \{w_1, w_2, \dots\}$ is a non-empty finite set of states. It is suitable to think of W as a set of historically accessible indices, i.e. $\{m/h \mid h \in H_m\}$ for a unique moment m in BT+AC frames.
- $Choice_i : W \to 2^{2^W}$ is a neighbourhood function, defined for every agent. $Choice_i(w)$ is a collection of sets, whose every element represents a set possible outcomes of i's specific action, available for her at w.
  - · Every function $Choice_i$ is monotonic (closed under supersets), does not contain an empty set and contains W itself, i.e. for arbitrary $w \in W, i \in Ags, X, Y \subseteq W : (X \in Choice_i(w) \wedge X \subseteq Y) \to Y \in Choice_i(w)$; for all $i \in Ags, w \in W$: $\emptyset \notin Choice_i(w), W \in Choice_i(w)$.
  - · $Choice_i \downharpoonleft (w)$ denotes a *non-monotonic core* of neighbourhood $Choice_i(w)$, i.e. a set of $\subseteq$-minimal elements of $Choice_i(w)$. $Choice_i \downharpoonleft (w)$ represents all proper actions, available for i at w, without redundant weaker ones. In addition, it is assumed that the non-monotonic core satisfies the property (un): $\bigcup Choice_i \downharpoonleft (w) = W$. Informally, it says that there is no historically possible state, which could not be an outcome of some proper action of an agent.
  - · Another crucial property is *independence of agents (ind)*: for all $a, b \in Ags$, $X, Y \subseteq W$, $w \in W$: $(X \in Choice_a(w) \wedge Y \in Choice_b(w)) \to X \cap Y \neq \emptyset$. Or, interchangeably, $X \in Choice_a(w) \to W \setminus X \notin Choice_b(w)$. The property states that every choice of actions is consistent: there is no way for one agent to take an action, such that another agent would be deprived of some of her choices.
  - · Agents' abilities are historically necessary, i.e. neighbourhoods stay the same over all states *(nec)*: $\forall w, w' \in W \forall X \subseteq W : X \in Choice_i(w) \to X \in Choice_i(w')$ for any agent $i \in Ags$.

The one-shot strategic stit model $\mathscr{M} = \langle \mathscr{F}, V \rangle$ extends osstit frame with a standard valuation function $V : Var \to 2^W$.

Definition 2.2 Neighbourhood strategic stit semantics
$\mathscr{M}, w \models p \Leftrightarrow w \in V(p)$
$\mathscr{M}, w \models \neg\phi \Leftrightarrow \mathscr{M}, w \not\models \phi$
$\mathscr{M}, w \models \phi \vee \psi \Leftrightarrow \mathscr{M}, w \models \phi$ or $\mathscr{M}, w \models \psi$
$\mathscr{M}, w \models \Box\phi \Leftrightarrow \forall w' \in W(\mathscr{M}, w' \models \phi)$
$\mathscr{M}, w \models [i]\phi \Leftrightarrow [\![\phi]\!] \in Choice_i(w)$
$\mathscr{M}, w \models [\exists_i]\phi \Leftrightarrow \forall X \in Choice_i \downharpoonright (w) \forall w' \in X(\mathscr{M}, w' \models \phi)$

As usual, $[\![\phi]\!]$ abbreviates $\{w \in W \mid \mathscr{M}, w \models \phi\}$.

2.2 Axioms for $\mathscr{L}_{osstit}$

(PL)	All tautologies of classical propositional logic
(S5$\Box$)	S5 for $\Box$ modality
(S5$[i]$)	S5 logic for $[\exists_i]$
(Incl)	$\Box\phi \to [i]\phi$
(M)	$[i](\phi \wedge \psi) \to ([i]\phi \wedge [i]\psi)$
(N)	$[i]\top$
(D)	$\neg[i]\bot$
(Pos)	$\Box\phi \equiv [\exists_i]\phi$
(Nec-A)	$[i]\phi \to \Box[i]\phi$
(Ind)	$[1]\phi_1 \wedge [2]\phi_2 \wedge \ldots \wedge [n]\phi_n \to \Diamond(\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n)$
(RE)	From $\phi \equiv \psi$, infer $[i]\phi \equiv [i]\psi$
(MP)	From $\phi, \phi \to \psi$, infer ψ

Table 1
Axioms for $\mathscr{L}_{osstit}$

Theorem 2.3 (Completeness) *Let* $\mathbf{C}$ *be a class of all osstit frames, corresponding to Definition 2.1. Let* $\mathbf{C} \models \phi$ *stand for* $\mathscr{F} \models \phi$ *for every* $\mathscr{F} \in \mathbf{C}$*. As usual,* $\Sigma \models_{\mathbf{C}} \phi$ *means that* ϕ *is a semantic consequence of some set of formulas* Σ *in all* $\mathbf{C}$*-frames. Then, for arbitrary* $\Gamma \subseteq \mathbf{L}_{osstit}, \phi \in \mathbf{L}_{osstit}$*, the following holds:*

$$\Gamma \models_{\mathbf{C}} \phi \Leftrightarrow \Gamma \vdash_{\mathscr{L}_{osstit}} \phi$$

Proof. *The proof may be obtained by a standard canonical model construction.* $\Box$

It is not hard to show that every BT+AC frame may be presented as a disjoint union of the one-shot strategic stit frames as defined earlier. The only crucial difference between BT+AC frames and osstit frames is the fact that while BT+AC *Choice* function returns a partition of a set H_m, neighbourhood functions $Choice \downharpoonright$ does not partition W: some pairs of $Choice_i \downharpoonright (w)$ elements may have non-empty-intersections. Nevertheless, the property which states that neighborhoods are pairwise disjoint is not modally definable for monotonic modal logic, which can be demonstrated via a bounded morphism technique.

3 Relations with other logics of ability and further research

We have proposed a fragment of classical stit logic, $\mathscr{L}_{osstit}$, with a non-normal strategic modality $[i]\phi$. The neighbourhood semantics for the fragment was presented, as well as soundness and strong completeness of $\mathscr{L}_{osstit}$ w.r.t. corresponding class of neighbourhood frames. The latter was obtained by standard method of canonical model construction.

The logic obtained may be seen as a special case of Coalition Logic with universal modality and without proper coalitions [8], i.e. with only such $\langle [C] \rangle$ operators, where C is either an empty-set or a singleton. Our logic describes such cases where agent's action profile is a partition of the set of all possible states, what corresponds to basic philosophical intuitions about agency of stit theory.

It is interesting to consider epistemic extensions of $\mathscr{L}_{osstit}$. It will allow us to reason about epistemic abilities: agent may be able to *knowingly* see to it that ϕ, i.e. she may be aware of the potential result of an action she is able to execute. One possibility is to construct such extension by adding a new modality for epistemic ability. It is worth investigating if all of $[i]$ properties should hold for the epistemic ability modality and what properties should be added to correctly display our intuitions about actions and knowledge interplay.

Besides that, the computational issues left untouched. It is known that the general group stit (i.e. allowing expressions of the form $[stit]_\Gamma \phi$, where $\Gamma \subseteq Ags$) without time operators is neither decidable nor finitely axiomatizable in case $|Ags| > 3$ [5]. It is also known that SAT problem for classical atemporal stit with single-agent modalities is NEXPTIME-complete if $|Ags| > 2$ [1]. Since $\mathscr{L}_{osstit}$ may be seen as a fragment of the latter, SAT complexity for it might show better computational behaviour.

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Cantor Derivative Logic in Topological Dynamics

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Abstract

Topological semantics for modal logic based on the Cantor derivative operator gives rise to derivative logics, also referred to as d-logics. Unlike logics based on the topological closure operator, d-logics have not previously been studied in the framework of dynamical systems, which are pairs (X, f) consisting of a topological space X equipped with a continuous function $f\colon X \to X$. We introduce the logics **wK4C**, **K4C** and **GLC** and show that they all have the finite Kripke model property and are sound and complete with respect to the d-semantics in this dynamical setting. We also prove a general result for the case where f is a homeomorphism, which in particular yields soundness and completeness for the corresponding systems **wK4H**, **K4H** and **GLH**. Of special interest is **GLC**, which is the d-logic of all dynamic topological systems based on a scattered space. We use the completeness of **GLC** and the properties of scattered spaces to exhibit the first sound and complete dynamic topological logic in the original trimodal language. In particular, we show that the version of **DTL** based on the class of scattered spaces is finitely axiomatisable over the original language, and that the natural axiomatisation is sound and complete.

Keywords: Dynamic topological logic, topological semantics, cantor derivative, dynamical systems.

1 Introduction

Dynamic (topological) systems are mathematical models of processes that may be iterated indefinitely. Formally, they are defined as pairs $\langle \mathfrak{X}, f\rangle$ consisting of a topological space $\mathfrak{X} = \langle X, \tau\rangle$ and a continuous function $f\colon X \to X$; the intuition is that points in the space $\mathfrak{X}$ 'move' along their orbit, $x, f(x), f^2(x), \ldots$ which usually simulates changes in time. *Dynamic topological logic* (**DTL**) combines modal logic and its topological semantics with linear temporal logic (see Pnueli [18]) in order to reason about dynamical systems in a decidable framework.

Due to their rather broad definition, dynamical systems are routinely used in many pure and applied sciences, including computer science. Such applications raise a need for effective formal reasoning about topological dynamics. Here, we may take a cue from modal logic and its topological semantics. The study of the latter dates back to McKinsey and Tarski [16], who proved that the modal logic **S4** is complete for a wide class of spaces, including the real line. Artemov, Davoren and Nerode [2] extended **S4** with a 'next' operator in the spirit of **LTL**, producing the logic **S4C**. They proved that this logic is sound and complete with respect to the class of all dynamic topological systems. The system **S4C** was enriched with the 'henceforth' tense by Kremer and Mints, who dubbed the new logic *dynamic topological logic* (**DTL**). Later, Konev et al. [14] showed that **DTL** is undecidable, and Fernández-Duque [9] showed that it is not finitely axiomatisable on the class of all dynamic topological spaces.

The aforementioned work on dynamic topological logic interprets the modal operator $\Diamond$ as a closure operator. However, McKinsey and Tarski had already contemplated semantics that are instead based on the Cantor derivative [16]: the *Cantor derivative* of a set A, usually denoted by $d(A)$, is the set of points x such that x is in the closure of $A \setminus \{x\}$. This interpretation is often called *d-semantics* and the resulting logics are called *d-logics.* These logics were first studied in detail by Esakia, who showed that the d-logic **wK4** is sound and complete with respect to the class of all topological spaces [7]. It is well-known that semantics based on the Cantor derivative are more expressive than semantics based on the topological closure. For example, consider the property of a space $\mathfrak{X}$ being *dense-in-itself*, meaning that $\mathfrak{X}$ has no isolated points. The property of being dense-in-itself cannot be expressed in terms of the closure operator, but it *can* be expressed in topological d-semantics by the formula $\Diamond\top$.

Logics based on the Cantor derivative appear to be a natural choice for reasoning about dynamical systems. However, there are no established results of completeness for such logics in the setting of dynamical systems, i.e. when a topological space is equipped with a continuous function. Our goal is to prove the finite Kripke model property, completeness and decidability of logics with the Cantor derivative operator and the 'next' operator $\bullet$ over some prominent classes of dynamical systems: namely, those based on arbitrary spaces, on T_D spaces (spaces validating the 4 axiom $\Box p \to \Box\Box p$) and on *scattered spaces.* Scattered spaces are topological spaces where every non-empty subspace has an isolated point. The reason for considering scattered spaces is to circumvent the lack of finite axiomatisability of **DTL** by restricting to a suitable subclass of all dynamical systems. In the study of dynamical systems and topological modal logic, one often works with *dense-in-themselves* spaces. This is a sensible consideration when modelling physical spaces, as Euclidean spaces are dense-in-themselves. However, some technical issues that arise when studying **DTL** over the class of all spaces disappear when restricting our attention to scattered spaces, which in contrast have many isolated points. Further, we consider dynamical systems where f is a *homeomorphism,* i.e. where f^{-1} is also a continuous function. Such dynamical systems are called *invertible.*

The basic dynamic d-logic we consider is **wK4C**, which consists of **wK4** and the temporal axioms for the continuous function f. In addition, we investigate two extensions of **wK4C**: **K4C** and **GLC**. As we will see, **K4C** is the d-logic of all T_D dynamical systems, and **GLC** is the d-logic of all dynamical systems based on a scattered space. Scattered spaces have gathered attention lately in the context of computational logic, as they may be used to model provability in formal theories [1], leading to applications in characterising their provably total computable functions [4]. Modal logic on scattered spaces enjoys definable fixed points [19], connecting it to the topological μ-calculus [3]. The latter is particularly relevant to us, as the expressive power gained by topological fixed points, including the tangled operators of $\mathbf{DTL}^*$, is absent in this setting. As the logic of scattered spaces is the Gödel-Löb modal logic **GL**, we refer to the dynamic topological logic of scattered spaces as *dynamic Gödel-Löb logic* (**DGL**).

Our goal is to demonstrate that the standard finite axiomatisation of **DGL** is sound and complete, leading to the first complete trimodal dynamic topological logic, as well as the first such logic combining the Cantor derivative with the infinitary 'henceforth' from **LTL**. By the 'standard axioms' we refer to the combination of the well-known axiomatisation of **GL** with **LTL** axioms for the tenses and $(\CIRCLE p \wedge \CIRCLE \Box p) \to \Box \CIRCLE p$ – a variant of the continuity axiom of Artemov et al. adapted for the Cantor derivative. The proof of completeness employs various advanced techniques from modal logic, including an application of Kruskal's theorem in the spirit of the work of Gabelaia et al. [13].

This paper summarises the results of Yoàv Montacute's Master's Thesis [17] and the consequent results it yielded [10,12].

2 Dynamic topological logic with the Cantor derivative

Given a non-empty set PV of propositional variables, the language $\mathcal{L}^{\circ}_{\Diamond}$ is defined recursively as follows:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \CIRCLE\varphi \mid \blacklozenge\varphi,$$

where $p \in \mathsf{PV}$. It consists of the Boolean connectives $\wedge$ and $\neg$, the temporal modalities 'next' $\CIRCLE$ and 'eventually' $\blacklozenge$ with its dual 'henceforth' $\blacksquare := \neg\blacklozenge\neg$, and the spatial modality $\Diamond$ for the Cantor derivative with its dual the co-derivative $\Box := \neg\Diamond\neg$. We define other connectives (e.g. $\vee$, $\to$) in the usual way.

Definition 2.1 A *dynamic topological model* is a tuple $\mathfrak{M} = (X, \tau, f, \nu)$, where (X, τ, f) is a dynamic topological system and $\nu : \mathsf{PV} \to \wp(X)$ is a *valuation function.* Given $\varphi \in \mathcal{L}^{\CIRCLE\blacklozenge}_{\Diamond}$, we define the *truth set* $[\![\varphi]\!] \subseteq X$ of a formula φ as follows:

- $[\![p]\!] = \nu(p)$;
- $[\![\neg\varphi]\!] = X \backslash [\![\varphi]\!]$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$;
- $[\![\Diamond\varphi]\!] = d([\![\varphi]\!])$;
- $[\![\CIRCLE\varphi]\!] = f^{-1}([\![\varphi]\!])$;
- $[\![\blacklozenge\varphi]\!] = \bigcup_{n\geq 0} f^{-n}([\![\varphi]\!])$.

Let us list the axiom schemes and rules that we will consider in this paper:

Taut := All propositional tautologies

K := $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

T := $\Box\varphi \to \varphi$

w4 := $\varphi \wedge \Box\varphi \to \Box\Box\varphi$

L := $\Box(\Box\varphi \to \varphi) \to \Box\varphi$

4 := $\Box\varphi \to \Box\Box\varphi$

$\mathrm{Next}_\neg$:= $\neg\bullet\varphi \leftrightarrow \bullet\neg\varphi$

$\mathrm{Next}_\wedge$:= $\bullet(\varphi \wedge \psi) \leftrightarrow \bullet\varphi \wedge \bullet\psi$

C := $\bullet\varphi \wedge \bullet\Box\varphi \to \Box\bullet\varphi$

H := $\Box\bullet\varphi \leftrightarrow \bullet\Box\varphi$

MP := $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

$\mathrm{Nec}_\Box$:= $\dfrac{\varphi}{\Box\varphi}$

$\mathrm{Nec}_\bullet$:= $\dfrac{\varphi}{\bullet\varphi}$

$\mathrm{K}_\blacksquare$:= $\blacksquare(\varphi \to \psi) \to (\blacksquare\varphi \to \blacksquare\psi)$

$\mathrm{Fix}_\blacksquare$:= $\blacksquare\varphi \to (\varphi \wedge \bullet\blacksquare\varphi)$

$\mathrm{Ind}_\blacksquare$:= $\blacksquare(\varphi \to \bullet\varphi) \to (\varphi \to \blacksquare\varphi)$

We define **wK4** := **K**+w4, **K4** := **K**+4, **S4** := **K4**+T and **GL** := **K4**+L. These logics are well known and characterise different classes of topological spaces. In addition, for a logic Λ over $\mathcal{L}_\Diamond$, $\Lambda\mathbf{F}$ is the logic over $\mathcal{L}_\Diamond^\bullet$ given by $\Lambda\mathbf{F} := \Lambda + \mathrm{Next}_\neg + \mathrm{Next}_\wedge + \mathrm{Nec}_\bullet$. This simply adds axioms of linear temporal logic to Λ, which hold whenever $\bullet$ is interpreted using a function. We define $\Lambda\mathbf{C} := \Lambda\mathbf{F}$+C and $\Lambda\mathbf{H} := \Lambda\mathbf{F}$+H, which as we will see correspond to derivative spaces with a continuous function or a homeomorphism respectively. Finally, let $\mathbf{DGL} := \mathbf{GLC} + \mathrm{K}_\blacksquare + \mathrm{Fix}_\blacksquare + \mathrm{Ind}_\blacksquare$.

The logic **K4** includes the axiom $\Box p \to \Box\Box p$, which is not valid over the class of all topological spaces. The class of spaces satisfying this axiom is denoted by T_D, defined as the class of spaces in which every singleton is the result of an intersection between an open set and a closed set. Moreover, Esakia showed that this is the logic of transitive derivative frames [8].

Many familiar topological spaces, including Euclidean spaces, satisfy the T_D property, making **K4** central in the study of topological modal logic. A somewhat more unusual class of spaces, which is nevertheless of particular interest to us, is the class of *scattered spaces.*

Definition 2.2 A topological space $\langle X, \tau\rangle$ is *scattered* if for every $S \subseteq X$, $S \subseteq d(S)$ implies $S = \varnothing$.

This is equivalent to the more common definition of a scattered space where a topological space is called scattered if every non-empty subset has an isolated point. Scattered spaces are closely related to converse well-founded relations.

Lemma 2.3 *If $\langle W, \sqsubset\rangle$ is an irreflexive frame, then $\langle W, \tau_\sqsubset\rangle$ is scattered iff $\sqsubset$ is converse well-founded.*

Theorem 2.4 (Simmons [20] and Esakia [6]) **GL** *is the logic of all scattered topological derivative spaces, as well as the logic of all converse well-founded derivative frames and the logic of all finite, transitive, irreflexive derivative frames.*

Aside from its topological interpretation, the logic **GL** is of particular interest as it is also the logic of provability in Peano arithmetic, as was shown by

Boolos [5]. Meanwhile, logics with the C and H axioms correspond to classes of dynamical systems.

Theorem 2.5 ([17,10])

(i) **wK4C(H)** *is sound and complete for*
  (a) *The class of all finite (invertible) dynamic* **wK4** *frames.*
  (b) *The class of all finite (invertible) dynamic topological systems with Cantor derivative.*

(ii) **K4C(H)** *is sound and complete for*
  (a) *The class of all finite (invertible) dynamic* **K4** *frames.*
  (b) *The class of all (invertible), T_D dynamic topological systems with Cantor derivative.*

(iii) **GLC(H)** *is sound and complete for*
  (a) *The class of all finite (invertible) dynamic* **GL** *frames.*
  (b) *The class of all finite (invertible), scattered dynamic topological systems with Cantor derivative.*

Mints suggested adding the 'henceforth' operator, $\blacksquare$, from Pnueli's linear temporal logic (**LTL**) [18], leading to a trimodal system was named *dynamic topological logic* (**DTL**). Kremer and Mints offered an axiomatisation for **DTL**, but Fernández-Duque proved that it is incomplete; in fact, **DTL** is not finitely axiomatisable [9]. Fernández-Duque also showed that **DTL** enjoys a natural axiomatisation when extended with the *tangled closure* [11].

Definition 2.6 Let $\langle X, d\rangle$ be a derivative space and let $\mathcal{S} \subseteq \wp(X)$. Given $A \subseteq X$, we say that $\mathcal{S}$ is tangled in A if for all $S \in \mathcal{S}$, $A \subseteq d(S \cap A)$. We define the *tangled derivative* of $\mathcal{S}$ as

$$\mathcal{S}^* := \bigcup\{A \subseteq X : \mathcal{S} \text{ is tangled in } A\}.$$

Fernández-Duque's axiomatisation is based on the extended language with the tangled operator $\diamond\!\!\!\!\diamond$.

Definition 2.7 For every model $\mathfrak{M}$, $\|\diamond\!\!\!\!\diamond\{\varphi_1, \ldots, \varphi_n\}\| = \{\|\varphi_1\|, \ldots, \|\varphi_n\|\}^*$.

Unlike the complete axiomatisation of **DTL** that requires the tangled operator, in the case of **DGL**, we are able to avoid this and use the original spatial operator $\Diamond$ alone. This is due to the following:

Theorem 2.8 *Let $\mathfrak{X} = \langle X, \tau\rangle$ be a scattered space and $\{\varphi_1, \ldots, \varphi_n\}$ a set of formulas. Then*

$$\diamond\!\!\!\!\diamond\{\varphi_1, \ldots, \varphi_n\} \equiv \bot.$$

Given this with the finite model theory for **GLC**, we can use an adapted version of the axiomatic system of Kremer and Mints [15] in order to provide a finite axiomatisation for **DGL**.

Theorem 2.9 ([12]) *Every formula $\varphi \in \mathcal{L}_{\Diamond}^{\bullet\blacklozenge}$ is valid on the class of scattered dynamical system iff it is derivable in* **DGL**.

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On strictly positive fragments of modal logics with confluence

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Abstract

We axiomatize strictly positive fragments of modal logics with the confluence axiom such as **K**.**2**, **D**.**2**, **D4**.**2** and **S4**.**2**. We show that the impact of the confluence axiom on the axiomatisation of strictly-positive fragments is rather weak. In presence of $\top \to \Diamond\top$ it simply disappears and does not contribute to axiomatisation. Without $\top \to \Diamond\top$ it gives rise to a weaker formula $\Diamond\top \to \Diamond\Diamond\top$.

Keywords: Modal logic, strictly positive logics, confluence

1 Introduction

In this paper we investigate strictly positive fragments of modal logics that include the axiom $\Diamond\Box p \to \Box\Diamond p$ corresponding to confluence. Strictly positive fragments consist of implications between strictly positive formulas. Strictly positive logics were studied in context of universal algebra [7], knowledge representation [9,3] and proof theory [1,5,2].

Confluence axiom is an example of simple but very useful formula. It appears in very different areas of modal logic ranging from epistemic logic to the logic of space-time and the logic of forcing. A variant of this axiom for two modalities is valid in any product of two Kripke frames and it is studied in context of multidimensional modal logic.

The question whether, given a strictly-positive implication ϕ, $K^+ + \phi$ axiomatises $SPF(\mathbf{K} + \phi)$ was thoroughly investigated in [8]. For example, this is true for $\phi = p \to \Diamond p$ and $\Diamond\Diamond p \to \Diamond p$ but not for $\Diamond p \to p$. The confluence axiom $\Diamond\Box p \to \Box\Diamond p$ cannot be rewritten as a strictly-positive implication. This rises a question how this axiom impacts the strictly-positive fragments of the logics with it. This question is by no means trivial, for example, Svyatlovskii showed in [10] that the strictly-positive fragment of **K4.3** is axiomatised by

$\Diamond\Diamond p \to \Diamond p$ and $\Diamond(p \wedge \Diamond q) \wedge \Diamond(p \wedge \Diamond r) \to \Diamond(p \wedge \Diamond q \wedge \Diamond r)$, which is a rather unexpected transformation of .3 axiom (also undefinable as an strictly-positive implication, see Section 9.1 of [8]).

In this paper we show that the impact of the confluence axiom on the axiomatisation of strictly-positive fragments is rather weak. In presence of $\top \to \Diamond\top$ it simply disappears and does not contribute to axiomatisation. Without $\top \to \Diamond\top$ it gives rise to a weaker formula $\Diamond\top \to \Diamond\Diamond\top$. Some may find it unsurprising, but in our opinion this is a remarkable property of the unimodal setting. In contrast, the strictly positive fragment of (2-modal) $\mathbf{K} + \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$ is *not* axiomatised by $(\Diamond_1 p) \wedge (\Diamond_2 q) \to \Diamond_1(p \wedge \Diamond_2\top) \wedge \Diamond_2(q \wedge \Diamond_1\top)$.

2 Preliminaries

2.1 Basic Modal Logic

Let $PV = \{p_1, p_2, \dots\}$ be a countable set of proposition letters, with typical members denoted by p, q, etc. *Modal formulas* over PV are built using the constants $\top$ and $\bot$, dual modal operators $\Diamond$ and $\Box$ and (classical) binary connectives $\vee$ and $\wedge$ and $\to$.

A *normal modal logic* is a set L of formulas that contains all tautologies, the formula $\Box(p \to q) \to (\Box p \to \Box q)$ and that is closed under the standard rules: Modus ponens, Uniform substitution, and Generalization (given ϕ infer $\Box\phi$). For a modal formula ϕ, $\mathbf{K} + \phi$ denotes that the smallest normal modal logic containing ϕ.

As usual, a Kripke frame is a pair $F = (W, R)$, where W is a non-empty set of worlds and R is a binary relation on W (that is $R \subseteq W \times W$). Sometimes we refer to the W- and R-components of $Frame$ as $Frame.W$ and $Frame.R$. A point u in W is called *final* in F if u has no R-successors. A (Kripke) model based on F is a pair $M = (F, V)$, where V is a function assigning to each proposition letter p a subset $V(p)$ of W. The inductive definition of the truth value of a formula ϕ at a point x in a model M is standard. The fact that ϕ is true at x in M is denoted by $M, x \models \phi$. In particular, boolean connectives are computed by classical truth tables within a point, $M, x \models \Diamond\phi$ if there is a point $y \in R(x)$ such that $M, y \models \phi$ and $M, x \models \Box\phi$ if for all points y such that $(x, y) \in R$ we have $M, y \models \phi$.

A formula ϕ is said to be *true in a model* $M = (W, R, V)$, in symbols $M \models \phi$, if ϕ is true at all worlds in W; ϕ is valid in a frame F, in symbols $F \models \phi$, if ϕ is true in all models based on F.

Each class of Kripke frames $\mathbf{C}$ gives rise to a normal modal logic $\mathrm{Log}(\mathbf{C}) = \{\phi \mid F \models \phi \text{ for all } F \text{ in } \mathbf{C}\}$. It is known that $\mathbf{K} + \Diamond\Box p \to \Box\Diamond p$ is the logic of all Kripke frames satisfying $Conf = \{\forall x \forall y \forall z\, (R(x, y) \wedge R(x, z) \to \exists v(R(y, v) \wedge R(z, v)))\}$.

2.2 Strictly Positive Implications

A strictly positive term (or SP-term) is a modal formula constructed from propositional variables, constants $\top$ and $\bot$, conjunction $\wedge$, and the unary diamond operator $\Diamond$. An SP-implication takes the form $\sigma \to \tau$, where σ and τ

are SP-terms. An SP-logic is a set of SP-implications which contains formulas

$$p \to p, \qquad p \to \top, \qquad p \wedge q \to q \wedge p, \qquad p \wedge q \to p, \tag{1}$$

and is closed under uniform substitution (of sp-terms for propositinal variables) and rules

$$\frac{\sigma \to \tau \quad \tau \to \varrho}{\sigma \to \varrho}, \qquad \frac{\sigma \to \tau \quad \sigma \to \varrho}{\sigma \to \tau \wedge \varrho}, \qquad \frac{\sigma \to \tau}{\Diamond\sigma \to \Diamond\tau} \tag{2}$$

(see also the Reflection Calculus **RC** of [1,5]). For an sp-implication ϕ, $K^+ + \phi$ denotes the smallest SP-logic containing ϕ.

For a normal modal logic **L** *the strictly positive fragment of* **L** is the following

$$SPF(\mathbf{L}) = \{\phi \mid \phi \text{ is an sp-implication and } \phi \in \mathbf{L}\}.$$

It is easy to check that $SPF(\mathbf{L})$ is an SP-logic.

Given an sp-term ρ, we define by induction a Kripke model $M_\rho = (T_\rho, V_\rho)$ based on a finite tree $T_\rho = (W_\rho, R_\rho)$ with root r_ρ. For $\rho = \top$, T_ρ consists of a single irreflexive point r_ρ with $V_\rho(p) = \emptyset$ for all variables p. For $\rho = p$, T_ρ consists of a single irreflexive point r_ρ, $V_\rho(p) = \{r_\rho\}$, and $V_\rho(q) = \emptyset$ for $q \neq p$. For $\rho = \rho_1 \wedge \rho_2$, we first construct disjoint M_{ρ_1} and M_{ρ_2}, and then merge their roots r_{ρ_1} and r_{ρ_2} into r such that $r \in V_\rho(q)$ iff $r_i \in V_{\rho_i}(q)$, for some $i = 1, 2$. Finally, for $\rho = \Diamond\rho'$, we add a fresh point r to $W_{\rho'}$, and set $R_\rho = R_{\rho'} \cup \{(r_\rho, r_{\rho'})\}$ and $V_\rho(p) = V_{\rho'}(p)$ for all variables p. We refer to M_ρ as the ρ-tree model.

Proposition 2.1 *For any sp-formula ρ, Kripke model M and point w in M, we have $M, w \models \rho$ iff there is a homomorphism $h : M_\rho \to M$ with $h(r_\rho) = w$.*

2.3 The Chase

A tuple-generating dependency is a first-order formula of the form

$$\forall\bar{x}\forall\bar{y}(\phi(\bar{x}, \bar{y}) \to \exists\bar{z}\psi(\bar{x}, \bar{z}), \tag{3}$$

where $\bar{x}$, $\bar{y}$ and $\bar{z}$ are disjoint tuples of variables and ϕ and ψ are possibly empty conjunction of R-atoms in respective sets of variables. Examples of TGDs include $Conf$ and and $Conf^+ = \{\forall x\forall y(R(x, y) \to \exists z\exists v(R(x, z) \wedge R(z, v)))\}$.

For such a conjunction χ we introduce constants c_v for object variables v and set $\Delta_\chi = (W^\chi, R^\chi)$ where $W^\chi = \{c_v \mid v \text{ occurs in } \chi\}$ and $R^\chi = \{(c_u, c_v) \mid R(u, v) \text{ is a conjunct of } \chi\}$.

Given a Kripke frame $F = (W, R)$ a trigger h for a TGD of the form (3) is a homomorphism from Δ_ϕ into F. A trigger h is *good* if h can't be extended to a homomorphism from Δ_ψ into F. An application of a good trigger h for $F = (W, R)$ to F is a frame $F' = (W', R')$ where W' extends W with *fresh* constants c_u for all $u \in \bar{z}$ and R' extends R with pairs $(f(u), f(v))$ for all atoms $R(u, v)$ in ψ where

$$f(u) = \begin{cases} h(u) & \text{if } u \text{ is in } \bar{x}, and \\ c_u & \text{if } u \text{ is in } \bar{z}. \end{cases}$$

For a set of TGDs Π by $ChaseStep(F, \Pi)$ we mean the relational structure which is the result of simultaneous application of all good triggers for F for all TGDs in Π to F. We define $Chase(F, \Pi)$ as the union or the inverse limit of the infinite chain $F \to ChaseStep(F, \Pi) \to ChaseStep(ChaseStep(F, \Pi)) \to \dots$.

It should be clear that $Chase(F, \Pi)$ always satisfies Π. For a Kripke model $M = (F, V)$ we define $Chase(M, \Pi)$ as $(Chase(F, \Pi), V)$. Those points of $Chase(F, \Pi)$ that are already in F are called *non-anonymous* and those that aren't are called *anonymous*.

Proposition 2.2 *For any SP-implication $s \to t$ we have $(s \to t) \in Log\{F \mid F \models \Pi\}$ iff $Chase(M_s, \Pi), root \models t$.*

($\Rightarrow$) Suppose that $Chase(M_s, \Pi), root \not\models t$. Clearly $Chase(M_s, \Pi), root \models s$ and $Chase(M_s, \Pi) \models \Pi$, so there exists a frame $F = Chase(M_s, \Pi)$ such that $F \models \Pi$, $F \not\models s \to t$. So this F refutes $s \to t$ showing that $s \to t$ is not in $Log\{F \mid F \models \Pi\}$.

($\Leftarrow$) Suppose that $Chase(M_s, \Pi), root \models t$. Consider arbitrary Kripke frame F satisfying Π, valuation θ and its point w such that $F, \theta, w \models s$. Hence there is a homomorphism f from M_s into F sending the root of M_s to w. Now consider (potentially infinite) step-by-step construction of $Chase(M_s, \Pi)$. Following this process in a step-by-step manner and using the fact that F satisfies Π on each step, we extend f to a homomorphism h from $Chase(M_s, \Pi)$ to F. Since $Chase(M_s, \Pi), root \models t$, it follows that $F, \theta, w \models t$. This shows that $s \to t$ is in $Log\{F \mid F \models \Pi\}$.

3 Results

Theorem 3.1 *$SPF(\mathbf{K} + \Diamond\Box p \to \Box\Diamond p) = K^+ + \Diamond\top \to \Diamond\Diamond\top$.*

In addition to $\mathbf{K}$ we consider the following logics

$$\mathbf{D} = \mathbf{K} + \top \to \Diamond\top, \qquad \mathbf{D4} = \mathbf{D} + \Diamond\Diamond p \to \Diamond p$$
$$\mathbf{T} = \mathbf{K} + p \to \Diamond p, \qquad \mathbf{S4} = \mathbf{T} + \Diamond\Diamond p \to \Diamond p.$$

Theorem 3.2 *For L belong to the following set of logics $\{\mathbf{D}, \mathbf{T}, \mathbf{D4}, \mathbf{S4}\}$ we have $SPF(L + \Diamond\Box p \to \Box\Diamond p) = SPF(L)$.*

The proof of Theorem 3.1 is based on the following observation:

Lemma 3.3 *For frame F we can define a partial function succ on $Chase(F, Conf^+)$ such that*

(i) *its domain contains all non-final points of F and the image of succ.*

(ii) *if $succ(u) = v$, then $(u, v) \in Chase(F, Conf^+).R$.*

Proof. Each non-final points u of F has a successor v, which gives rise to a trigger h for $Conf^+$. If this trigger is good, we set $succ(u)$ to be c_z introduced by an application of this trigger, otherwise we set $succ(u)$ to be $h'(z)$ where h is an extension of h to the head of the rule. Then we define $succ$ on those points of $ChaseStep(F, Conf^+)$ where it has not been defined so far. Then we deal similarly with $ChaseStep(ChaseStep(F, Conf^+), Conf^+)$ and so on. $\Box$

Proof. [Proof of Theorem 3.1.] It should be clear that every theorem of $K^+ + \Diamond\top \to \Diamond\Diamond\top$ is a strictly-positive theorem of $\mathbf{K} + \Diamond\Box p \to \Box\Diamond p$ since $\Diamond\top \to \Diamond\Diamond\top$ is a theorem of $\mathbf{K} + \Diamond\Box p \to \Box\Diamond p$. Now take a strictly positive implication $s \to t$ such that $s \to t \in \mathbf{K} + \Diamond\Box p \to \Box\Diamond p$. By Proposition 2.2 it follows that $Chase(M_s, Conf), root \models t$. Therefore there exists a rooted homomorphism h from M_t into $Chase(M_s, Conf)$. If M_s is a singleton, then both $Chase(M_s, Conf)$ and $Chase(M_s, Conf^+)$ are isomorphic to M_s and we are done. Otherwise note that given a propositional variable p, its valuation $V(p)$ in $Chase(M_s, Conf)$ does not contain anonymous points of $Chase(M_s, \Pi)$ (P1) and that every generated submodel rooted at an anonymous point of $Chase(M_s, Conf)$ contains only anonymous points (P2).

It follows that $Chase(M_s, Conf^+), root \models t$. Indeed, we can define a homomorphism h' from M_t into $Chase(M_s, \Pi')$ by recursion. We set $h'(u) = h(u)$ for non-anonymous $h(u)$. For anonymous $h(u)$ we look at the parent v of u in M_s and set $h'(u) = succ(h'(v))$ assuming that $h'(v)$ is already defined. It should be clear that h' is a homomorphism. $\Box$

The proof of Theorem 3.2 is similar. We use TGDs $Ser = \{\forall x(\top \to \exists y\, R(x,y))\}$, $Refl = \{\forall x(\top \to R(x,x))\}$ and $Trans = \{\forall x \forall y \forall x(R(x,y) \wedge R(y,x) \to R(x,z))\}$ and the fact that properties (P1) and (P2) still hold in the setting of L. For example, for $s \to t \in \mathbf{S4.2}$, (P1) and (P2) hold for $Chase(M_s, Ser \cup Trans \cup Refl \cup Conf)$, and this allows us to transform a homomorphism $h : M_t \to Chase(M_s, Ser \cup Trans \cup Refl \cup Conf)$ into one $h' : M_t \to Chase(M_s, Ser \cup Trans \cup Refl)$.

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The Dependence Problem in Varieties of Modal Semilattices

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Abstract

For a variety $\mathcal{V}$, we call terms $t_1, \ldots, t_n$ $\mathcal{V}$-dependent if there is a non-valid equation in $y_1, \ldots, y_n$ that becomes valid in $\mathcal{V}$ when the y_i are substituted by the t_i. The dependence problem is the problem of deciding whether any finite number of terms are $\mathcal{V}$-dependent. We show that the dependence problem for the varieties of modal join-semilattices and modal meet-semilattices is decidable.

Keywords: Dependence, Uniform Interpolation, Modal Logic, Modal Semilattices.

1 Introduction

In [1], de Jongh and Chagrova introduced the notion of dependence for intuitionistic propositional logic **IPC**. Formulas $\varphi_1, \ldots, \varphi_n$ are called **IPC**-*dependent* if there exists a formula $\psi(p_1, \ldots, p_n)$ such that $\vdash_{\mathsf{IPC}} \psi(\varphi_1, \ldots, \varphi_n)$ and $\nvdash_{\mathsf{IPC}} \psi(p_1, \ldots, p_n)$. Using Pitt's proof of uniform interpolation for **IPC**, they showed that the **IPC**-dependence of any finite number of **IPC**-formulas is decidable.

We can study this concept in any logic. In particular, for the modal logic **K**, the proof in [1] cannot be applied, but Lutz and Wolter showed for the description logic **ALC** that if right uniform deductive interpolants exist, they can be calculated [3]. This result yields the decidability of **K**-dependence of any finite number of **K**-formulas. Note that the decidability of this problem for **K** does not imply the decidability of the same problem in fragments of **K**. If it is decidable in some fragments of **K**, a similar approach might provide an alternative proof for the decidability of dependence in **K**. Since the result in [3] is stronger than required, this approach could yield a more elegant proof.

We consider two of these fragments algebraically, namely the varieties of modal join-semilattices $\mathcal{MJS}$ and modal meet-semilattices $\mathcal{MMS}$. We show that for these structures, it is decidable whether any finite number of terms are dependent. Structures like these are considered for example by Kikot et al [2]. Finally, we consider the subvarieties of $\mathcal{MJS}$ and $\mathcal{MMS}$ defined by the additional axiom $x \leq \Box x$ and show that the dependence of finitely many terms is decidable therein.

2 Definitions and Basic Results

Let $\mathcal{V}$ be a variety, then we call terms $t_1, \ldots, t_n \in \mathrm{Tm}(\bar{x})$ *$\mathcal{V}$-dependent* if for some equation $\varepsilon \in \mathrm{Eq}(y_1, \ldots, y_n)$,

$$\vDash_{\mathcal{V}} \varepsilon(t_1, \ldots, t_n) \qquad \text{and} \qquad \nvDash_{\mathcal{V}} \varepsilon;$$

otherwise we call $t_1, \ldots, t_n$ *$\mathcal{V}$-independent*. The problem of deciding whether any finite set of terms are $\mathcal{V}$-dependent is called the *dependence-problem for $\mathcal{V}$*. This notion of $\mathcal{V}$-dependence is a special case of an algebraic notion of dependence introduced by Marcewski in [4].

Let $\Gamma, \Delta \subseteq \mathrm{Eq}(\bar{y})$. We write $\Gamma \mathrel{|\!\!\!\sim}_{\mathcal{V}} \Delta$ if for any substitution $\sigma \colon \mathrm{Tm}(\bar{y}) \to \mathrm{Tm}(\omega)$ extended to equations by setting $\sigma(s \approx t) = \sigma(s) \approx \sigma(t)$,

$$\vDash_{\mathcal{V}} \sigma(\Gamma) \quad \Longrightarrow \quad \vDash_{\mathcal{V}} \sigma(\delta) \text{ for some } \delta \in \Delta.$$

We say that a set $\Delta \in \mathrm{Eq}(\bar{y})$ is *$\mathcal{V}$-refuting* for $\bar{y}$ if for any equation $\varepsilon(\bar{y})$,

$$\nvDash_{\mathcal{V}} \varepsilon \quad \Longleftrightarrow \quad \{\varepsilon\} \mathrel{|\!\!\!\sim}_{\mathcal{V}} \Delta.$$

Lemma 2.1 *For any $\mathcal{V}$-refuting set Δ for $\{y_1, \ldots, y_n\}$, the terms $t_1, \ldots, t_n \in \mathrm{Tm}(\bar{x})$ are $\mathcal{V}$-dependent if and only if $\vDash_{\mathcal{V}} \delta(t_1, \ldots, t_n)$ for some $\delta \in \Delta$.*

Thus, for varieties that have a decidable equational theory and for which a finite $\mathcal{V}$-refuting set for any finite $\bar{y}$ can be constructed, the dependence-problem is decidable.

Note that for varieties with a lattice-order, we can consider inequations instead of equations.

Example 2.2 We define $[n] \coloneqq \{1, \ldots, n\}$. Consider $\mathcal{L}at$, the variety of all lattices, and let

$$\Delta_n \coloneqq \Big\{ y_i \leq \bigvee_{j \in [n] \setminus \{i\}} y_j \mid i \in [n] \Big\} \cup \Big\{ \bigwedge_{j \in [n] \setminus \{i\}} y_j \leq y_i \mid i \in [n] \Big\}.$$

In [5], we show that Δ_n is a $\mathcal{L}at$-refuting set for $\{y_1, \ldots, y_n\}$ and thus, the dependence problem for $\mathcal{L}at$ is decidable.

3 Modal Join-Semilattices

We consider $\mathcal{MJS}$, the variety of $\langle A, \vee, \Box \rangle$-algebras such that $\langle A, \vee \rangle$ is a semilattice and

$$\Box a \vee \Box b \leq \Box(a \vee b), \quad \text{for all } a, b \in A.$$

The method used to prove the decidability of the dependence problem for $\mathcal{L}at$ cannot be applied here, but we can use a similar approach.

Lemma 3.1 *The following set of $\mathcal{MJS}$-inequations in $\bar{y}$ is $\mathcal{MJS}$-refuting for $\bar{y}$:*

$$\begin{aligned} \Delta_{\bar{y}} \coloneqq \{ y \leq s \mid y \in \bar{y} \text{ and } s \neq s_1 \vee y \vee s_2 \} \\ \cup \{ \Box^k y \leq y' \mid y, y' \in \bar{y} \text{ and } k \geq 1 \}. \end{aligned}$$

For any $\delta \in \Delta_{\bar{y}}$, we have $\nvDash_{\mathcal{MJS}} \delta$. Let $\varepsilon \in \mathrm{Eq}(\bar{y})$ with $\nvDash_{\mathcal{MJS}} \varepsilon$. We describe a procedure to obtain a finite $\Sigma_\varepsilon \subseteq \Delta_{\bar{y}}$ such that for any substitution σ, whenever $\vDash_{\mathcal{MJS}} \sigma(\varepsilon)$, also $\vDash_{\mathcal{MJS}} \sigma(\delta)$ for some $\delta \in \Sigma_\varepsilon$. We demonstrate this procedure by considering an example. Let $\delta = \Box^2 y_1 \leq \Box(y_2 \vee \Box y_3)$. δ is non-valid and $\Box y_1 \leq y_2 \vee \Box y_3$ is also non-valid. Suppose there is a substitution σ, such that $\vDash_{\mathcal{MJS}} \Box^2\sigma(y_1) \leq \Box(\sigma(y_2) \vee \Box(y_3))$. This yields $\vDash_{\mathcal{MJS}} \Box\sigma(y_1) \leq \sigma(y_2) \vee \Box\sigma(y_3)$ and then either $\vDash_{\mathcal{MJS}} \sigma(y_1) \leq \sigma(y_3)$ or $\vDash_{\mathcal{MJS}} \sigma(y_1) \leq s_2$, where $\sigma(y_2) = s_1 \vee \Box s_2 \vee s_3$. In the second case, we also get $\vDash_{\mathcal{MJS}} \Box\sigma(y_1) \leq \Box s_2 \leq s_1 \vee \Box s_2 \vee s_3 = \sigma(y_2)$. The inequations $\Box y_1 \leq y_2$ and $y_1 \leq y_3$ are non-valid and in $\Delta_{\bar{y}}$. Therefore the set $\{\Box y_1 \leq y_2, y_1 \leq y_3\}$ has the required form.

The set $\Delta_{\bar{y}}$ is obviously infinite, therefore the previous lemma does not yield the decidability of the dependence-problem for $\mathcal{MJS}$. However, depending on the terms considered, we can restrict the set described in Lemma 3.1 to obtain a finite set that can be used to determine whether these terms are $\mathcal{MJS}$-dependent. For a term t, we denote the modal depth of t by $\mathrm{md}(t)$.

Theorem 3.2 *Let $t_1, \ldots, t_n \in \mathrm{Tm}(\bar{x})$ and let $\bar{y} = \{y_1, \ldots, y_n\}$. Then $t_1, \ldots, t_n$ are $\mathcal{MJS}$-dependent if and only if there is an inequation $\delta \in \Delta^d_{\bar{y}}$ such that*

$$\vDash_{\mathcal{MJS}} \delta(t_1, \ldots, t_n),$$

where $\Delta^d_{\bar{y}} \coloneqq \{\delta \in \Delta_{\bar{y}} \mid \mathrm{md}(\delta) \leq d\}$ and $d \coloneqq \max\{\mathrm{md}(t_1), \ldots, \mathrm{md}(t_n)\}$.

Corollary 3.3 *The dependence problem for $\mathcal{MJS}$ is decidable.*

Furthermore, we consider the variety $\mathcal{MJS}_{\mathrm{refl}}$ of modal join-semilattices satisfying for all algebras $\langle A, \wedge, \Box\rangle$ and all $a \in A$

$$a \leq \Box a.$$

For any term $s \in \mathrm{Tm}(\bar{y})$, distributing all the boxes over the joins by inductively applying $\Box a \vee \Box b \leq \Box(a \vee b)$ for $a, b \in \mathrm{Tm}(\bar{y})$, yields a term s' of the form $\bigvee_{i=1}^{l} \Box^{k_i} y_{1i}$, satisfying

$$\vDash_{\mathcal{MJS}_{\mathrm{refl}}} s' \leq s.$$

If y occurs in s, then there is some $j \in \{1, \ldots, l\}$ such that $y_{1j} = y$, and we get

$$\vDash_{\mathcal{MJS}_{\mathrm{refl}}} y \leq \Box^{k_j} y \quad \Longrightarrow \quad \vDash_{\mathcal{MJS}_{\mathrm{refl}}} y \leq \bigvee_{i=1}^{l} \Box^{k_i} y_{1i} \quad \Longrightarrow \quad \vDash_{\mathcal{MJS}_{\mathrm{refl}}} y \leq s.$$

Then we can show that the following set of $\mathcal{MJS}_{\mathrm{refl}}$-inequations in $\bar{y}$ is $\mathcal{MJS}_{\mathrm{refl}}$-refuting for $\bar{y}$:

$$\begin{aligned} \Delta_{\bar{y}} \coloneqq \{&y \leq s \mid y \text{ does not occur in } s\} \\ &\cup \{\Box^k y \leq y' \mid y, y' \in \bar{y},\, y \neq y' \text{ and } k \geq 0\}. \end{aligned}$$

And again, the terms $t_1, \dots, t_n$ are $\mathcal{MJS}_{\text{refl}}$-dependent if and only if $\vDash_{\mathcal{MJS}_{\text{refl}}} \delta(t_1, \dots, t_n)$, for some $\delta \in \Delta^d_{\bar{y}}$, where d and $\Delta^d_{\bar{y}}$ are defined in the same way as in Theorem 3.2.

Therefore, the dependence problem for $\mathcal{MJS}_{\text{refl}}$ is decidable.

4 Modal Meet-Semilattices

We consider $\mathcal{MMS}$, the variety of $\langle A, \wedge, \Box\rangle$-algebras such that $\langle A, \wedge\rangle$ is a semilattice and

$$\Box a \wedge \Box b = \Box(a \wedge b), \quad \text{for all } a, b \in A.$$

Let $\mathcal{P}_{fin}(\mathbb{N})$ be the set of all finite subsets of $\mathbb{N}$ and define the operation $\Box$ on $\mathcal{P}_{fin}(\mathbb{N})$ for any $a_1, \dots, a_k \in \mathbb{N}$ as follows:

$$\Box\{a_1, \dots, a_k\} \coloneqq \{a_1 + 1, \dots, a_k + 1\}.$$

Proposition 4.1 *The free $\mathcal{MMS}$-algebra over $\bar{x} = \{x_1, \dots, x_m\}$ is isomorphic to*

$$\mathbf{PN}_m \coloneqq \langle \mathrm{PN}_m, \cup, \Box\rangle,$$

where $\mathrm{PN}_m \coloneqq \mathcal{P}_{fin}(\mathbb{N})^m \backslash \{\langle \varnothing, \dots, \varnothing\rangle\}$, *and* $\cup$, $\Box$ *are defined component-wise.*

This proposition can be proved by first showing that $\mathbf{PN}_m$ is in $\mathcal{MMS}$ for each $m \in \mathbb{N}_{>0}$ and then proving that the map

$$\alpha : \{x_1, \dots, x_m\} \to \mathrm{PN}_m; \quad x_i \mapsto \langle \varnothing, \dots, \underset{i}{\{0\}}, \dots, \varnothing\rangle,$$

can be extended to the bijective homomorphism $\beta : \mathbf{F}(\bar{x}) \to \mathbf{PN}_m$. Using this isomorphism, we can study the valid inequations in $\mathcal{MMS}$ more easily. Let $\beta([s])_i$ denote the i-th component of $\beta([s])$, where $[s]$ is an element of $\mathbf{F}(\bar{x})$. Then it follows that

$$\begin{aligned} \vDash_{\mathcal{MMS}} s \leq r \quad &\iff \quad \vDash_{\mathcal{MMS}} s \wedge r \approx s \\ &\iff \quad \beta([s \wedge r]) = \beta([s]) \cup \beta([r]) = \beta([s]) \\ &\iff \quad \beta([r])_i \subseteq \beta([s])_i, \text{ for all } i \in \{1, \dots, m\}. \end{aligned}$$

Let $t_1, \dots, t_n$ be some terms in $\bar{x}$. If for a variable x_i, $[t] = [\bigwedge_{j=1}^{l} \Box^{a_j} x_i \wedge t']$ where the a_j are all distinct and t' does not contain x_i, then we can find $k, k_2, \dots, k_l \in \mathbb{N}$ such that $k \neq k_j$ for all $j \in \{2, \dots, l\}$ and

$$\Box^k \beta([t]) \subseteq \beta([t]) \cup \bigcup_{j=2}^{l} \Box^{k_j} \beta([t]).$$

Similarly, if a variable x_i occurs in both t and t', then we can either find k' such that $\beta([t]) \subseteq \Box^{k'} \beta([t'])$, or we can find k such that $\Box^k \beta([t]) \subseteq \beta([t'])$. Note that $y \wedge \bigwedge_{j=2}^{l} \Box^{k_j} y \leq \Box^k y$, $y \leq \Box^{k'} y'$ and $\Box^k y \leq y'$ are non-valid inequations. These ideas can be used to prove the following theorem.

Theorem 4.2 *Let $t_1, \ldots, t_n$ be some $\mathcal{MMS}$-terms in $x_1, \ldots, x_m$. Then the following are equivalent:*

(i) *$t_1, \ldots, t_n$ are $\mathcal{MMS}$-dependent;*

(ii) *There is a term $t \in \{t_1, \ldots, t_n\}$, such that for each variable x occurring in t one of the following holds:*
   (a) *$[t] = [\Box^a x \wedge \Box^b x \wedge t']$, for $a, b \in \mathbb{N}$ with $a \neq b$.*
   (b) *There is a term $t_x \in \{t_1, \ldots, t_n\} \backslash \{t\}$ such that x also occurs in t_x.*

Corollary 4.3 *The dependence problem for $\mathcal{MMS}$ is decidable.*

Finally, we consider the variety $\mathcal{MMS}_{\mathrm{refl}}$ of modal meet-semilattices satisfying for all algebras $\langle A, \wedge, \Box \rangle$ and all $a \in A$

$$a \leq \Box a.$$

We define the operation $\Box$ on $\mathbb{N}$ for $a \in \mathbb{N}$ as follows:

$$\Box a = a + 1.$$

Then, similarly as for $\mathcal{MMS}$, we can describe the free $\mathcal{MMS}_{\mathrm{refl}}$-algebra over $\{x_1, \ldots, x_m\}$ concretely:

$$\mathbf{F}(x_1, \ldots, x_m) \cong \langle \mathbb{N}^m \setminus \{\langle 0, \ldots, 0 \rangle\}, \min, \Box \rangle,$$

where min and $\Box$ are defined component-wise. And analogously to Theorem 4.2, terms $t_1, \ldots, t_n$ in $x_1, \ldots, x_m$ are $\mathcal{MMS}_{\mathrm{refl}}$-dependent if and only if there is $t \in \{t_1, \ldots, t_n\}$ such that for each variable occurring in t, there is a term $t_x \in \{t_1, \ldots, t_n\} \setminus \{t\}$ such that x also occurs in t_x.

Thus, the dependence problem for $\mathcal{MMS}_{\mathrm{refl}}$ is decidable.

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On decidable extensions of Propositional Dynamic Logic

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Abstract

We describe a family of decidable propositional dynamic logics, where atomic programs satisfy some extra conditions (for example, given by axioms of the logics K5, S5, or K45 for different atomic modalities). Using recent transfer results for logics enriched with modalities for the transitive closure and converse relations [6], [7], we show that many extensions of Propositional Dynamic Logic (with converses) is complete with respect to its standard finite models.

Keywords: Propositional Dynamic Logic, filtration, fusion of modal logics, finite model property, decidability

1 Definitions

Normal logics and Kripke semantics. Let At be a set of *atomic modalities*, $\mathrm{PV} = \{p_i \mid i < \omega\}$ a set of *propositional variables.* The *set of modal formulas* $\mathrm{Fm}(\mathrm{At})$ is generated by the following grammar:

$$\varphi ::= \bot \mid p \mid (\varphi \to \psi) \mid \Diamond\varphi \qquad (p \in \mathrm{PV},\ \Diamond \in \mathrm{At})$$

Other connectives are defined in the standard way; in particular, $\Box\varphi$ is $\neg\Diamond\neg\varphi$.

A *normal modal* At-*logic* is a set of formulas $L \subseteq \mathrm{Fm}(\mathrm{At})$ such that:

(i) L contains all Boolean tautologies;

(ii) For all $\Diamond \in \mathrm{At}$, $\Diamond\bot \leftrightarrow \bot \in L$ and $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q \in L$;

(iii) L is closed under the rules of Modus Ponens, uniform substitution, and *monotonicity*: $\varphi \to \psi \in L$ implies $\Diamond\varphi \to \Diamond\psi \in L$ for all $\Diamond \in \mathrm{At}$.

For an At-logic L and a set Ψ of At-formulas, $L + \Psi$ is the least normal modal At-logic that contains $L \cup \Psi$.

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An At-frame is a structure $F = (W, (R_\Diamond)_{\Diamond \in \mathrm{At}})$, where each $R_\Diamond$ is a binary relation on W. A model on an At-frame is a structure $M = (F, \vartheta)$, where $\vartheta : \mathrm{PV} \to 2^W$. The truth definition is standard:

- $M, x \models p_i$ iff $x \in \vartheta(p_i)$
- $M, x \not\models \bot$
- $M, x \models \varphi \to \psi$ iff either $M, x \not\models \varphi_i$ or $M, x \models \psi$.
- $M, x \models \Diamond\varphi$ iff there exists y such that $xR_\Diamond y$ and $M, y \models \varphi$

We set $M \models \varphi$ iff $M, x \models \varphi$ for all $x \in M$, $F \vDash \varphi$ iff $M \vDash \varphi$ for all M based on F; $Log(F)$ is the set $\{\varphi \in \mathrm{Fm}(\mathrm{At}) \mid F \vDash \varphi\}$. For a class $\mathcal{F}$ of frames, $Log(\mathcal{F}) = \bigcap\{Log(F) \mid F \in \mathcal{F}\}$. A logic L is *Kripke complete* iff $L = Log(\mathcal{F})$ for a class $\mathcal{F}$ of frames. A logic L has the *finite model property (fmp)* iff $L = Log(\mathcal{F})$ for a class $\mathcal{F}$ of finite frames.

For a logic L, $\mathrm{Mod}(L)$ is the class of models such that $M \models L$.

Propositional Dynamic Logics. Let At be finite. The set Prog(At) (*"programs"*) is generated by the following grammar:

$$\langle e\rangle, \langle f\rangle ::= \Diamond \mid \langle e \cup f\rangle \mid \langle e \circ f\rangle \mid \langle e^+\rangle \qquad (\Diamond \in At,\ e, f \in \mathrm{Prog(At)})$$

Remark 1.1 Our language of programs is test-free.

Definition 1.2 A *normal propositional dynamic* At-*logic* is a normal Prog(At)-logic that contains the following formulas for all $\langle e\rangle, \langle f\rangle \in \mathrm{Prog(At)}$:

A1 $\langle e \cup f\rangle p \leftrightarrow \langle e\rangle p \vee \langle f\rangle p$,

A2 $\langle e \circ f\rangle p \leftrightarrow \langle e\rangle\langle f\rangle p$,

A3 $\langle e\rangle p \to \langle e^+\rangle p$,

A4 $\langle e\rangle\langle e^+\rangle p \to \langle e^+\rangle p$,

A5 $\langle e^+\rangle p \to \langle e\rangle p \vee \langle e^+\rangle(\neg p \wedge \langle e\rangle p)$.

The least normal propositional dynamic At-logic is denoted by $\mathbf{PDL}(\mathrm{At})$.

We also consider dynamic logics with converses. The set $\mathrm{Prog}_t(\mathrm{At})$ is given by the following grammar:

$$\langle e\rangle, \langle f\rangle ::= \Diamond \mid \langle e \cup f\rangle \mid \langle e \circ f\rangle \mid \langle e^+\rangle \mid \langle e^{-1}\rangle \qquad (\Diamond \in At,\ e, f \in \mathrm{Prog}_t(\mathrm{At}))$$

A *normal propositional dynamic* At-*logic with converses* is a normal $\mathrm{Prog}_t(\mathrm{At})$-logic that for all $\langle e\rangle, \langle f\rangle \in \mathrm{Prog}_t(\mathrm{At})$ contains the formulas **A1**–**A5** and the formulas

A6 $p \to [e]\langle e^{-1}\rangle p$

A7 $p \to [e^{-1}]\langle e\rangle p$

The least dynamic At-logic with converses is denoted by $\mathbf{PDL}_t(\mathrm{At})$.

The validity of formulas **A1-A7** in a frame $(W, (R_\Diamond)_{\Diamond \in \mathrm{Prog}_t(\mathrm{At})})$ is equivalent to the following identities:

$$R_{\langle e \circ f\rangle} = R_{\langle e\rangle} \circ R_{\langle f\rangle},\ R_{\langle e \cup f\rangle} = R_{\langle e\rangle} \cup R_{\langle f\rangle},\ R_{\langle e^+\rangle} = (R_{\langle e\rangle})^+,\ R_{\langle e^{-1}\rangle} = (R_{\langle e\rangle})^{-1},$$

where R^+ denotes the transitive closure of R, R^{-1} the converse of R; models based of such frames are called *standard*; see, e.g., [5, Chapter 10]. It is known that $\mathbf{PDL}_t(\mathrm{At})$ is complete with respect to its standard finite models [10]. Our aim is to extend this result to a family of extensions of $\mathbf{PDL}_t(\mathrm{At})$.

2 Filtrations and decidable extensions of dynamic logic

Let $\varphi \in \mathrm{Fm}(\mathrm{At})$, then $\mathrm{Sub}(\varphi)$ is the set of all subformulas of φ. A set of formulas Γ is Sub-closed if $\varphi \in \Gamma$ implies $\mathrm{Sub}(\varphi) \subseteq \Gamma$.

For a model $M = (W, (R_s)_{s\in\mathrm{At}}, \vartheta)$ and a set of formulas Γ,

$$x \sim_\Gamma y \text{ iff } \forall \psi \in \Gamma\ (M, x \models \psi \text{ iff } M, y \models \psi).$$

Definition 2.1 Let Γ be a Sub-closed set of formulas. A filtration of a model $M = (W, (R_\Diamond)_{\Diamond\in\mathrm{At}}, \vartheta)$ through Γ (or Γ-filtration, for short) is a model $\widehat{M} = (\widehat{W}, (\widehat{R_\Diamond})_{\Diamond\in\mathrm{At}}, \hat{\theta})$ s.t.

(i) $\widehat{W} = W/{\sim}$ for some equivalence relation $\sim$ such that $\sim \subseteq \sim_\Gamma$, i.e.,

$$x \sim y \text{ implies } \forall \psi \in \Gamma\ (M, x \models \psi \Leftrightarrow M, y \models \psi).$$

(ii) $\widehat{M}, [x] \models p$ iff $M, x \models p$ for all $p \in \Gamma$. Here $[x]$ is the class of x modulo $\sim$.

(iii) For all $\Diamond \in \mathrm{At}$, we have $R_{\Diamond\sim} \subseteq \widehat{R_\Diamond} \subseteq R_{\Diamond\sim}^{\Gamma}$, where

$$\begin{aligned} &[x]\, R_{\Diamond\sim}\, [y] \text{ iff } \exists x' \sim x\ \exists y' \sim y\ (x'\, R_\Diamond\, y') \\ &[x]\, R_{\Diamond\sim}^{\Gamma}\, [y] \text{ iff } \forall \psi\ (\Diamond\psi \in \Gamma \,\&\, M, y \models \psi \Rightarrow M, x \models \Diamond\psi) \end{aligned}$$

The relations $R_{\Diamond\sim}$ and $R_{\Diamond\sim}^{\Gamma}$ on $\widehat{W}$ are called the *minimal* and the *maximal filtered relations*, respectively.

If $\sim\, = \,\sim_\Delta$ for some finite sub-closed set of formulas $\Delta \supseteq \Gamma$, then $\widehat{M}$ is called a *definable* Γ*-filtration* of the model M. If $\sim\, = \,\sim_\Gamma$, the filtration $\widehat{M}$ is said to be *strict*.

The following fact is well-known, see, e.g., [2]:

Lemma 2.2 (Filtration lemma) *Suppose that* Γ *is a finite* Sub*-closed set of formulas and* $\widehat{M}$ *is a* Γ*-filtration of a model* M*. Then, for all points* $x \in W$ *and all formulas* $\varphi \in \Gamma$*, we have:*

$$M, x \models \varphi \textit{ iff } \widehat{M}, [x] \models \varphi.$$

Definition 2.3 We say that a class of models $\mathcal{M}$ *admits definable (strict) filtration* iff for any finite Sub-closed set of formulas Γ and any $M \in \mathcal{M}$, there is a finite model in $\mathcal{M}$ that is a definable (strict) Γ-filtration of M. A logic *admits definable (strict) filtration* iff the class $\mathrm{Mod}(L)$ of its models does.

It is immediate from the Filtration lemma that if a logic admits filtration, then it is complete with respect to the class of its finite models, and consequently, to the class of its finite frames.

Strict filtrations are the most widespread in the literature; it is well-known that the logics $\mathrm{K}, \mathrm{T}, \mathrm{K4}, \mathrm{S4}, \mathrm{S5}$ admit strict filtration, see e.g., [2]. A wide family of logics that admit strict filtration is the *stable logics* [1]. Constructions where the initial equivalence is refined, also have been used since late 1960s [11], [4], and later, see, e.g., [13]. Refining the initial equivalence makes the filtration method much more flexible. For example, the logics $\mathrm{K5} = \mathrm{K} + \{\Diamond p \to \Box\Diamond p\}$ and $\mathrm{K} + \{\Diamond\Diamond\Diamond p \to \Diamond p\}$ do not admit strict filtration, but admit definable filtration. For the logic $\mathrm{K} + \{\Diamond\Diamond\Diamond p \to \Diamond p\}$, definable filtrations were constructed in [4]. That K5 admits definable filtration follows from the fact that this logic is locally finite [9] and Theorem 2.6 below.

In [6] and [7], filtrations were used to obtain a number of transfer results for logics enriched with modalities for the transitive closure and converse relations. The following theorem is a corollary of [7, Theorem 4.6] and the proof of [6, Theorem 2.4].

Theorem 2.4 *Let* At *be finite. If an* At*-logic L admits definable filtration, then* $\mathbf{PDL}_t(\mathrm{At}) + L$ *has the fmp.*

Let $L_1, \ldots, L_n$ be logics (for some $n \leq \omega$) in languages that have mutually disjoint sets of modalities. The *fusion* $L_1 * \cdots * L_n$ is the smallest logic that contains $L_1, \ldots, L_n$. We adopt the following convention: for logics $L_1, \ldots, L_n$ in the same language, we also write $L_1 * \cdots * L_n$ assuming that we "shift" modalities; e.g., $\mathrm{K5} * \mathrm{K5}$ denotes the bimodal logic given by the two axioms $\Diamond_i p \to \Box_i \Diamond_i p$, $i = 1, 2$.

It is known that the fusion of consistent modal logics is a conservative extension of its components [14]. Also, the fusion operation preserves Kripke completeness, decidability, and the finite model property [3], [8], and [15].

In [7], it was noted that if the logics $L_1, \ldots, L_n$ admit strict filtration, then the fusion $L = L_1 * \ldots * L_n$ admits strict filtration; it follows from Theorem 2.4 that $\mathbf{PDL}_t(\mathrm{At}) + L$ has the fmp for the case of such L. This does not cover many important examples where (some of) L_i do not admit strict filtration (like in the case of the logic $\mathrm{K5} * \mathrm{K5}$). The following two theorem allows to significantly extend application of Theorem 2.4.

Theorem 2.5 *Let $k < \omega$. If logics L_i, $i < k$, admit definable filtration, then their fusion admits definable filtration.*

This theorem generalizes [7, Theorem 4.8].

Theorem 2.6 *If L is locally finite, then L admits definable filtration.*

The proof of this theorem is based on methods proposed in [12].

Corollary 2.7 *Let n and At be finite, $L_1, \ldots, L_n$ normal logics that admit definable filtration, and $L_1 * \cdots * L_n \subseteq \mathrm{Fm}(\mathrm{At})$. Then $\mathbf{PDL}_t(\mathrm{At}) + L_1 * \cdots * L_n$ has the fmp.*

Corollary 2.8 *Let n and At be finite, $L_1, \ldots, L_n$ normal logics, $L_1 * \cdots * L_n \subseteq \mathrm{Fm}(\mathrm{At})$. If each L_i is one of the logics* K, T, K4, K5, K45, S4, S5, $\mathrm{K} + \{p \to \Box\Diamond p\}$, $\mathrm{K} + \{\Diamond\top\}, \mathrm{K4} + \{\Diamond\top\}$, $\mathrm{K} + \Diamond^m p \to \Diamond p$ *for $m \geq 1$, or a locally finite logic, then $\mathbf{PDL}_t(\mathrm{At}) + L_1 * \cdots * L_n$ has the fmp. If also all L_i are finitely axiomatizable, then $\mathbf{PDL}_t(\mathrm{At}) + L_1 * \cdots * L_n$ is decidable.*

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Topological product of S4.1 x S4

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Abstract

We consider product of modal logics in topological semantics and prove that the topological product of S4.1 and S4 is the fusion of logics S4.1 and S4 plus one extra axiom. This is an example of a topological product of logics that is greater than the fusion but less than the semiproduct of the corresponding logics.

Keywords: Modal logic, topological semantics, product of modal logics, S4.1, S4

1 Introduction

The product of Kripke frames was defined and studied by many authors (cf. [7,11,6]). It is a natural way to combine modal logic and simulate two-dimensional structures.

There are many ways to combine modal logics. Some of them are syntactical and some semantical. The simplest syntactical combination is the fusion. The fusion of two unimodal logics L_1 and L_2 is the minimal bimodal logic containing axioms from L_1 rewritten with $\Box_1$ and axioms from L_2 rewritten with $\Box_2$. Notation: $\mathsf{L}_1 * \mathsf{L}_2$.

The product of two modal logics is a semantical way to combine logics. The product of two modal logics is the logic of the class of all products of structures of the corresponding logics. Such construction based on the Kripke frames was introduced by Shehtman in 1978 [15]. Later in 2006 van Benthem et al. [1] introduced a similar construction based on topological spaces [2].

Ph.Kremer in [8] proved that topological product of $\mathsf{S4}$ and $\mathsf{S5}$ is the semiproduct $\mathsf{S4} * \mathsf{S5} + \Box_1\Box_2 p \to \Box_2\Box_1 p + \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p = [\mathsf{S4}, \mathsf{S5}]^{EX}$. It is also known that the topological product of $\mathsf{S5}$ and $\mathsf{S5}$ is coincide with the Kripke product.

In this work we prove that topological product of $\mathsf{S4.1}$ and $\mathsf{S4}$ is strictly between the fusion and the semiproduct of the corresponding logics. We show

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[2] "Product of topological spaces" is a well-known notion in Topology but it is different from what we use here (for details see [1])

that this product is equal to $\mathsf{S4.1} * \mathsf{S4} + \Diamond_1\Box_2(\Diamond_1 p \to \Box_1 p)$. It is the first example if this kind with known axiomatization.

In 2020 at the AiML I announced that topological product of $\mathsf{S4.1}$ and $\mathsf{S4}$ is equal to the fusion $\mathsf{S4.1} * \mathsf{S4}$, but later I found a mistake in the proof. This short paper is correcting that mistake.

2 Definitions and the context

Let us start with propositional modal logics. Assume we have a countably infinite set of propositional letters PROP. A (modal) formula is defined recursively by using the Backus-Naur form as follows:

$$A ::= p \mid \bot \mid (A \to A) \mid \Box_i A,$$

where $p \in$ PROP is a propositional letter, and $\Box_i$ is a modal operator $(i = 1, \dots, N)$. Other connectives are introduced as abbreviations: classical connectives are expressed through $\bot$ and $\to$, and $\Diamond_i$ is a shortcut for $\neg\Box_i\neg$. The set of all modal formulas is denoted by $\mathcal{ML}_N$.

Definition 2.1 A *(normal modal) logic* is a set of modal formulas closed under Substitution $\left(\frac{A(p)}{A(B)}\right)$, Modus Ponens $\left(\frac{A,\, A\to B}{B}\right)$ and Generalization $\left(\frac{A}{\Box_i A}\right)$ rules, containing all the classical tautologies and the normality axioms:

$$\Box_i(p \to q) \to (\Box_i p \to \Box_i q).$$

$\mathsf{K_N}$ denotes *the minimal normal modal logic with n modalities* and $\mathsf{K} = \mathsf{K_1}$.

Let L be a logic and Γ be a set of formulas, then $\mathsf{L} + \Gamma$ denotes the minimal logic containing L and Γ. If $\Gamma = \{A\}$, then we write $\mathsf{L} + A$ rather than $\mathsf{L} + \{A\}$.

Logic $\mathsf{S4}$ is well known:

$$\mathsf{S4} = \mathsf{K} + \Box p \to p + \Box p \to \Box\Box p.$$

Where K is the minimal normal modal logic (see. [4]).

Definition 2.2 ([1]) Let $\mathfrak{X}_1 = (X_1, T_1)$ and $\mathfrak{X}_2 = (X_2, T_2)$ are two topological spaces. We define the (bitopological) product of them as bitopological space $\mathfrak{X}_1 \times \mathfrak{X}_2 = (X_1 \times X_2, T_1^h, T_2^v)$. Topology T_1^h is the topology with the base $\{U \times \{x_2\} \mid U \in T_1 \ \&\ x_2 \in X_2\}$ and topology T_2^v is the topology with the base $\{\{x_1\} \times U \mid x_1 \in X_1 \ \&\ U \in T_2\}$.

The topological product of two topologically complete modal logics L_1 and L_2 is the following logic with two modalities:

$$L_1 \times_t L_2 = Log(\{\mathfrak{X}_1 \times \mathfrak{X}_2 \mid \mathfrak{X}_1, \mathfrak{X}_2 \text{ — topological spaces}, \mathfrak{X}_1 \models L_1, \mathfrak{X}_2 \models L_2\})$$

Theorem 2.3 ([16]) $\mathsf{S4} \times_t \mathsf{S4} = \mathsf{S4} * \mathsf{S4}$.

Formula $A1 = \Box\Diamond p \to \Diamond\Box p$ is called the McKinsey axiom. It is also well-known and studied both in the Kripke semantics and in topological semantics.

In Kripke semantics this formula corresponds to the following property in the presence of $\mathsf{S4}$. For an $\mathsf{S4}$-frame $F = (W, R)$

$$F \models A1 \iff \forall w \in W \exists u \in W(wRu \wedge R(u) = \{u\}),$$

where $R(u) = \{t \mid uRt\}$. The proof is straightforward.

Definition 2.4 In topological space $\mathfrak{X}$ point x is *isolated* if set $\{x\}$ is open in $\mathfrak{X}$. $\mathfrak{X}$ is *weakly scattered* if the set of isolated points of $\mathfrak{X}$ is dense in $\mathfrak{X}$, that is if any open subset has an isolated point.

Topological semantics for logic $\mathsf{S4.1}$ was studied in [16,3,2]. Logic $\mathsf{S4.1}$ is the modal logic of the class of weakly scattered spaces.

Lemma 2.5 *Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be topological spaces and $\mathfrak{X}_1$ is weakly scattered. Then*

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models \Diamond_1 \Box_2 (\Diamond_1 p \to \Box_1 p).$$

Proof. Let us take $(x, y) \in \mathfrak{X}_1 \times \mathfrak{X}_2$ and a horizontal open neighborhood $U \times \{y\}$, where $U \in \tau_1(x)$ and τ_1 is the neighborhood function of $\mathfrak{X}_1$. Since $\mathfrak{X}_1$ is weakly scattered set U contain an isolated (in $\mathfrak{X}_1$) point x'.

It follows that for any $y' \in \mathfrak{X}_2$ point (x', y') is isolated in horizontal topology and hence

$$(x', y') \models \Diamond_1 p \to \Box_1 p.$$

This finishes the proof. $\Box$

Lemma 2.6 $\Diamond_1 \Box_2 (\Diamond_1 p \to \Box_1 p) \notin \mathsf{S4.1} * \mathsf{S4}$.

This lemma can be proved by providing an appropriate $\mathsf{S4.1} * \mathsf{S4}$-frame.

Corollary 2.7 $\mathsf{S4.1} \times_t \mathsf{S4} \neq \mathsf{S4.1} * \mathsf{S4}$.

Lemma 2.8 $\Diamond_1 \Box_2 (\Diamond_1 p \to \Box_1 p) \in [\mathsf{S4.1}, \mathsf{S4}]^{EX}$.

3 Main Result

Our main result is

Theorem 3.1 $\mathsf{S4.1} \times_t \mathsf{S4} = \mathsf{S4.1} * \mathsf{S4} + \Diamond_1 \Box_2 (\Diamond_1 p \to \Box_1 p)$.

Lemma 3.2 *For a Kripke frame $F = (W, R_1, R_2)$ such that $F \models \mathsf{S4.1} * \mathsf{S4}$*

$$F \models \Diamond_1 \Box_2 (\Diamond_1 p \to \Box_1 p) \iff \forall x \exists y (xR_1 y \;\&\; \forall z (yR_2 z \Rightarrow R_1(z) = \{z\})).$$

Lemma 3.3 *Logic $\mathsf{S4.1} * \mathsf{S4} + \Diamond_1 \Box_2 (\Diamond_1 p \to \Box_1 p)$ is canonical.*

Logic $\mathsf{S4.1} * \mathsf{S4}$ is canonical since all its axioms are canonical. The canonicity of $\Diamond_1 \Box_2 (\Diamond_1 p \to \Box_1 p)$ can be obtatianed by a straightforward modification of the proof of canonisity of McKinsey axiom (see [5, Ch. 5, Th. 5.21]).

By $\mathbb{T}_{2,2}$ we define a (2,2) branching infinite tree. $\mathbb{T}_{2,2} = (T_{2,2}, R_1, R_2)$, where $T_{2,2} = \{a_1, a_2, b_1, b_2\}^*$ — is the set of finite words in a 4-letter alphabet, for $\alpha, \beta \in T_{2,2}$

$$\alpha R_1 \beta \Leftrightarrow \beta = \alpha \cdot a_{i_1} \dots a_{i_k};$$
$$\alpha R_2 \beta \Leftrightarrow \beta = \alpha \cdot b_{j_1} \dots b_{j_l}.$$

By $\mathbb{T}_{2,2+2} = (W, R_1', R_2')$ we define a frame that we get by putting a copy of the infinte 2-tree $\mathbb{T}_2 = (T_2, R)$ above each point in $\mathbb{T}_{2,2}$ in the following way:

- $W = T_{2,2} \times \{\varnothing\} \cup T_{2,2} \times T_2$,
- $(\alpha, \varnothing) R_i' (\beta, \varnothing)$ iff $\alpha R_i \beta$,
- $(\alpha, \varnothing) R_1' (\alpha, \varepsilon)$, where ε is the empty word and the root of $\mathbb{T}_2$,
- $(\alpha, a) R_2' (\alpha, b)$ iff aRb for $a, b \in T_2$.

Lemma 3.4 $\mathsf{S4.1} * \mathsf{S4} + \Diamond_1 \Box_2 (\Diamond_1 p \to \Box_1 p) = Log(\mathbb{T}_{2,2+2})$.

To finish the proof of the main theorem we need to find a weakly scattered space $\mathfrak{X}$ and a space $\mathcal{Y}$ such that there exist an open and continuous surjection $f : \mathfrak{X} \times \mathcal{Y} \to Top_2(\mathbb{T}_{2,2+2})$, where $Top_2(\mathbb{T}_{2,2+2})$ is a topological space with two Alexandroff topologies based on R_1' and R_2'.

As $\mathcal{Y}$ we take $\mathcal{N}_\omega(T_2)$ — a topological space of pseudo-infinite paths with stops in T_2. We assume that $T_2 = \{1,2\}^*$ and R is the relation of being a prefix.

Definition 3.5 *A path with stops* on T_2 is a tuple $x_1 \dots x_n$, so that $x_i \in \{0,1,2\}$. For a path with stops w we define f_F by induction: $f_F(\varepsilon) = \varepsilon$; $f_F(w0) = f_F(w)$; $f_F(w1) = f_F(w)1$; and $f_F(w2) = f_F(w)2$.

A *pseudo-infinite paths (with stops)* is an infinite sequence of 0,1 and 2 with only finitely many non-zero elements. Let W_ω be the set of all pseudo-infinite paths in T_2.

In the following we define $f_F : W_\omega \to T_2$. Let $\alpha = x_1 \dots x_n \dots$ be a pseudo-infinite path then

$$\begin{aligned} st(\alpha) &= \min \{ N \mid \forall k > N (a_k = 0) \}; \\ \alpha|_k &= x_1 \dots x_k; \\ f_F(\alpha) &= f_F(\alpha|_{st\,\alpha}); \\ U_k(\alpha) &= \{ \beta \in W_\omega \mid \alpha|_m = \beta|_m \ \&\ f_F(\alpha) R f_F(\beta),\ m = \max(k, st(\alpha)) \}. \end{aligned}$$

Definition 3.6 Sets $U_n(\alpha)$ form a base for topology T and $\mathcal{N}_\omega(F) = (W_\omega, T)$.

This definition up to small changes follows the [9].

Lemma 3.7 ([9]) *There exists an open and continuous surjective map* $g : \mathcal{Y} \times \mathcal{Y} \to Top_2(\mathbb{T}_{2,2})$.

Technically in [9] instead of $\mathbb{T}_{2,2}$ we considered ω branching tree but in case of $\mathsf{S4}$ it does not matter.

And as $\mathfrak{X} = (\mathcal{Y} \times \mathbb{N}, T_X)$ where T_X is the topology with base including the following sets:

- $V_n(\alpha, 0) = (U_n(\alpha) \times \{0\}) \cup (\{\alpha\} \times \mathbb{N}_{\geq n})$, where $\mathbb{N}_{\geq n} = \{ i \mid i \geq n \}$;
- $V_n(\alpha, k) = \{(\alpha, k)\}$ for $k \geq 1$.

Lemma 3.8 $\mathfrak{X}$ *is weakly scattered.*

The proof is straightforward.

We define $f : \mathfrak{X} \times \mathcal{Y} \to \mathbb{T}_{2,2+2}$ as follows:

$$f((\alpha, 0), \beta) = (g(\alpha, \beta), \varnothing);$$
$$f((\alpha, k), \beta) = (g(\alpha|_k, \beta|_k), f_F(x_{k+1}x_{k+2}\ldots))\,, \text{ for } \beta = x_1x_2\ldots x_kx_{k+1}\ldots.$$

Lemma 3.9 *Function f is surjective, open and continuous.*

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Bounded ultrafilter extensions and ultrafilter unions are elementary

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Abstract

Our main purpose is the study of general first-order model theoretic properties of structures having their roots in modal logic. We show that certain notion of boundness in case of these structures results elementary equivalent ultrafilter extensions, and elementary equivalent ultrapowers with their ultrafilter unions. Also, we touch upon a case when the embeddings into the ultrafilter extensions and unions are elementary and a case when elementary substructures are lifted to their ultrafilter extensions.

Keywords: Ultrafilter extension, ultrafilter union, bounded degree, bounded path, elementary equivalence

1 Introduction

The various connections between first-order and modal logic has always been a central issue in the model theory of modal logics. Numerous questions (e.g. correspondence theory, axiomatizability, canonicity) were challenged by so many aspects of model theory and universal algebra. Nevertheless, one of the central notions of these investigations is the concept of ultrafilter extension, first introduced in [3] and has become a fundamental concept understanding modal logics. The construction has clear analogies with ultrapowers when proving certain results in modal logic, hence the motto: *"ultrafilter extensions in model theory of modal logics play the rôle similar to ultrapowers in first-order model theory"*.

Very much the same goes with ultrafilter unions that were introduced in [6] as a construction in the modal definability of models, hence one can formulate the motto *"ultrafilter unions in model theory of modal logics play the rôle similar to ultraproducts in first-order model theory"*.

Our purpose is to make this analogy more exact. It is quite natural to ask how the modal and first-order properties of a structure are connected to its ultrafilter extension or its ultrapower and similarly, how these properties of an ultraproduct are connected to the ultrafilter union. The literature mainly raises the question of *"what type of first-order formulas are preserved under taking ultrafilter extension?"* [1], the question which turns out to be Π^1_1-hard, hence disclaiming the existence of any characterization theorem [2]. Here we take the reverse approach and investigate *what type of structures are elementary equivalent to their ultrafilter extension* and what *type of ultraproducts are elementary equivalent to their ultrafilter unions.* We isolate such class of structures with the aid of certain notion of boundedness. Below we present some follow up results of an ongoing research without proofs.

2 Preliminaries, notations

Throughout we restrict ourselves to structures $\mathfrak{F} = \langle W, R\rangle$ for the basic propositional modal language $\mathcal{L}_\Diamond$, consisting of a single unary modality and its corresponding first-order frame language $\mathcal{L} = \{R, =\}$ of graph theory. The truth and validity of a formula in both languages is defined in the usual manner. We recall some terminology and definitions.

Definition 2.1 A structure $\mathfrak{F} = \langle W, R\rangle$ is of *bounded degree* if there is some $n \in \omega$ such that for all $w \in W$ we have $|R^+(w) \cup R^-(w)| \leq n$, where $R^+(w) = \{v : Rwv\}$ and $R^-(w) = \{v : Rvw\}$. The family of structures $\{\mathfrak{F}_i : i \in I\}$ is of *n-bounded degree*, if each $\mathfrak{F}_i$ is uniformly bounded by the degree n.

Definition 2.2 By $\widetilde{R}$ let us denote the relation $R \cup R^{-1}$. We say that two elements w and v are $\widetilde{R}$-connected if there is an $\widetilde{R}$-path from w to v, otherwise they are disconnected. The frame $\mathfrak{F}$ is of *bounded paths* if there is $n \in \omega$ such that the property

$$\forall x_0 \ldots \forall x_n (\bigwedge_{i=0}^{n-1} \widetilde{R}x_i x_{i+1} \to \bigvee_{i \neq j} x_i = x_j)$$

holds in $\mathfrak{F}$.

Definition 2.3 (Ultrafilter extension, ultrafilter union) The *ultrafiler extension* of $\mathfrak{F}$ is the structure $\mathfrak{F}^{\mathfrak{ue}} = \langle \mathrm{Uf}(W), R^{\mathfrak{ue}}\rangle$, where $\mathrm{Uf}(W)$ is the set of all ultrafilters over W and $R^{\mathfrak{ue}}$ is defined as:

$$R^{\mathfrak{ue}}uv \Leftrightarrow (\forall X \in v) R^-(X) \in u$$

Let $\{\mathfrak{F}_i : i \in I\}$ be a set of structures, D be an ultrafilter over I and $\mathfrak{F} = \biguplus \mathfrak{F}_i$. We say that $X \subseteq W$ is *D-compatible* if $\{i \in I : X \cap W_i \neq \emptyset\} \in D$. An $u \in \mathrm{Uf}(W)$ is D-compatible if for all $X \in u$ is D-comptible. The *ultrafilter union* of $\mathfrak{F}$ over D is the structure $\biguplus_D \mathfrak{F}_i = \langle W_D, R_D\rangle$, where

- W_D is the set of D-compatible ultrafilters over W,
- $R_D = R^{\mathfrak{ue}} \cap W_D$

We note that every $s \in \prod W_i$ determines a D-compatible ultrafilter for some fixed ultrafilter D over I, as for $s = \langle s_i : i \in I\rangle$ the set

$$\pi_D(s) = \{X \subseteq W : \{i \in I : s_i \in X_i\} \in D\}$$

is a D-compatible ultrafilter. Also the function $\eta_D : \prod \mathfrak{F}_i/D \to \biguplus_D \mathfrak{F}_i$, where

$$\eta_D([s]_D) = \pi_D(s)$$

is an embedding, moreover if D is the principal ultrafilter generated by $i \in I$, we have $\biguplus_D \mathfrak{F}_i \cong \mathfrak{F}_i^{\mathfrak{ue}}$.

Definition 2.4 (Neighbourhood) Let $w \in W$, for a fixed $n \in \omega$ we define the following set:

$\langle w \rangle^0 = \{w\}$

$\langle w \rangle^{i+1} = R^+(\langle w \rangle^i) \cup R^-(\langle w \rangle^i) \cup \langle w \rangle^i$ for $i \leq n$

The *n-neighbourhood* of w is defined to be the structure $\langle \langle w \rangle^n, R{\restriction}_{\langle w \rangle^n} \rangle$.

3 Main results and further work

3.1 Frames with bounded degree

Our first theorem presents a "Łoś Lemma-like" connection between frames of bounded degree and their ultrafilter extensions that plays a key rôle in our further investigations.

Theorem 3.1 *Let $\mathfrak{F}$ be a frame of bounded degree, then for all $n \in \omega$ and $u \in \mathrm{Uf}(W)$ we have:*

$$\{w \in W : \langle w \rangle^n \cong \langle u \rangle^n\} \in u$$

A detailed account of this result can be found in [4]. As a consequence, using Ehrenfeucht-Fraïssé game and n-bisimulation one gets the following results:

Theorem 3.2 *Let $\{\mathfrak{F}_i : i \in I\}$ be a family of n-bounded structures, D be an ultrafilter over I, then*

(i) $\mathsf{Th}(\prod \mathfrak{F}_i/D) = \mathsf{Th}(\biguplus_D \mathfrak{F}_i)$

(ii) $\Lambda(\prod \mathfrak{F}_i/D) = \Lambda(\biguplus_D \mathfrak{F}_i)$

Corollary 3.3 *For any frame $\mathfrak{F}$ and $\mathfrak{G}$ with bounded degree if $\mathfrak{F} \equiv \mathfrak{G}$, then $\mathfrak{F}^{\mathfrak{ue}} \equiv \mathfrak{G}^{\mathfrak{ue}}$.*

We can slightly improve the result above by assuming that D is a regular ultrafilter over I. The next lemma is essential to the forthcoming theorem.

Lemma 3.4 *Assume D is regular over I and $u \neq \pi_D(s)$ for all $s \in \prod W_i$. Then for all $n \in \omega$ there are $s_1, \ldots, s_n \in \prod W_i$ such that $\langle \pi_D(s_i) \rangle^\omega \cong \langle u \rangle^\omega$ for $i < n$.*

From this one can prove an important connection between ultraproducts and ultrafilter unions.

Theorem 3.5 *Let $\{\mathfrak{F}_i : i \in I\}$ be a family of n-bounded structures, D be regular over I. Then the embedding $\eta_D : \prod \mathfrak{F}_i/D \to \biguplus_D \mathfrak{F}_i$ is elementary.*

Choosing a saturated enough ultrapower of $\mathfrak{F}$ it is possible to prove the following statement.

Theorem 3.6 *For any structure $\mathfrak{F}$ with bounded degree there exists an ultrapower ${}^I\mathfrak{F}/D$ such that $\mathfrak{F} \preceq \mathfrak{F}^{\mathfrak{ue}} \preceq {}^I\mathfrak{F}/D$.*

[5] wonders about theories for which elementary substructures are lifted to their ultrafilter extensions. Here we have an immediate corollary:

Theorem 3.7 *Let $\mathfrak{F}$ and $\mathfrak{G}$ be graphs of bounded degree such that $f : \mathfrak{F} \to \mathfrak{G}$ is an elementary embedding. Then the function $f^{\mathrm{ue}} : \mathfrak{F}^{\mathrm{ue}} \to \mathfrak{G}^{\mathrm{ue}}$ defined by*

$$f^{\mathrm{ue}}(u) = \{Y : f[X] \subseteq Y \text{ for some } X \in u\}$$

is an elementary embedding.

In the general case when there is no restriction on the structures, one can prove a property analogous to [1] Lemma 15.18. Let u be any fixed individual variable. Recall that an $r(u)$-formula is obtained from atomic formulas of the forms Rux, Rxu, $x = u$ and atomic formulas in which u does not occur, by applying Boolean operators and restricted existential quantification of the form $\exists y(Ruy \wedge \varphi(y))$ or $\exists y(Ryu \wedge \varphi(y))$, where y does not occur in $\varphi(u)$.

Proposition 3.8 *Let $\{\mathfrak{F}_i : i \in I\}$ be a family of structures φ be an $r(u)$ formula and let $\mathfrak{F} = \biguplus \mathfrak{F}_i$, then*

$$\biguplus_F \mathfrak{F}_i \models \varphi(u, \pi_F(s_1), \dots, \pi_F(s_n)) \Leftrightarrow (\exists U \in u)(\forall i \in I)(\forall w_i \in U_i)\, \mathfrak{F} \models \varphi(w_i, s_1(i), \dots, s_n(i))$$

Proof. By induction on the construction of φ. $\Box$

As a consequence, the existential closure of an $r(u)$ formula is preserved from the ultraproduct to the ultrafitler union.

3.2 Frames with bounded paths

We shortly turn into structures with bounded paths and list some properties in connection with the first-order theory of their ultrafilter extensions. Before doing so we introduce some terminology.

By $X \subseteq_0 W$ we will mean a finite subset. Let $u \in \mathrm{Uf}(W)$ and $X \subseteq_0 \mathrm{Uf}(W)$. We define the formula $\varphi^u_X(x)$ as follows: for each $a_i \in X - \{u\}$ consider distinct variables x_i and formulas

$$Rx_ix_j \text{ iff } R^{\mathrm{ue}}a_ia_j \quad \text{for } a_i, a_j \in X \tag{1}$$
$$\neg Rx_ix_j \text{ iff } \neg R^{\mathrm{ue}}a_ia_j \quad \text{for } a_i, a_j \in X \tag{2}$$
$$Rxx_i \text{ iff } R^{\mathrm{ue}}ua_i \quad \text{for } a_i \in X \tag{3}$$
$$Rx_ix \text{ iff } R^{\mathrm{ue}}a_iu \quad \text{for } a_i \in X \tag{4}$$
$$\neg Rxx_i \text{ iff } \neg R^{\mathrm{ue}}ua_i \quad \text{for } a_i \in X \tag{5}$$
$$\neg Rx_ix \text{ iff } \neg R^{\mathrm{ue}}a_iu \quad \text{for } a_i \in X \tag{6}$$
$$Rxx \text{ iff } R^{\mathrm{ue}}uu \tag{7}$$
$$x_i \neq x_j \text{ and } x \neq x_i \tag{8}$$

Let $\psi(x, x_1, \dots, x_n)$ be be the conjunction of the formulas from (1)-(8) and define $\varphi^u_X(x) := \exists x_1, \dots \exists x_n \psi(x, x_1 \dots, x_n)$. In a very weak sense one can think of $\varphi^u_X(x)$ as a type of u over X.

For a fixed $\mathfrak{F}$ and $w \in W$ we say that a $X \subseteq_0 \langle w\rangle^\omega$ is a w-connected neighborhood of w if each $v \in X$ is $\widetilde{R}$-connected to w in $\langle w\rangle^X = \langle X, R{\restriction}_X\rangle$.

Proposition 3.9 *Let $\mathfrak{F}$ be a frame of bounded paths. If for any $v \in \mathrm{Uf}(W)$ and any v-connected Y the set*

$$U_v = \{w \in W : (\exists Y_v \subseteq_0 \langle w \rangle^\omega) \langle w \rangle^{Y_v} \cong \langle v \rangle^Y\} \in v$$

then for all $u \in \mathrm{Uf}(W)$ and $X \subseteq_0 \mathrm{Uf}(W)$ we have $\{w \in W : \mathfrak{F} \models \varphi_X^u(w)\} \in u$.

A similar "Łoś Lemma-like" theorem can be proved for structures with bounded paths.

Theorem 3.10 *Let $\mathfrak{F}$ be a frame of bounded paths. For all $u \in \mathrm{Uf}(W)$ and u-connected X the set*

$$U_u = \{w \in W : (\exists X_u \subseteq_0 \langle w \rangle^\omega) \langle w \rangle^{X_u} \cong \langle u \rangle^X\} \in u$$

We wonder if such a property can be used to prove that frames with bounded paths are elementary equivalent to their ultrafilter extensions. In particular we have the following problem:

Problem 3.11 *For each structure $\mathfrak{F}$ with bounded paths is it true that $\mathfrak{F} \equiv \mathfrak{F}^{\mathfrak{ue}}$?*

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The smallest modal system

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Abstract

Over the last decades logicians like Segerberg, Wansing and Humberstone started their investigations of modal logics with a very a weak system of modal logic. This system is usually formulated in the language of **CPL** containing one additional unary modal operator $\Box$, satisfies Modus Ponens, uniform substitution and contains all (classical) tautologies. In what follows, we will present a non-deterministic relational semantics for the so-called smallest modal system and briefly discuss some possible extensions.

Keywords: Smallest Modal Logic, Non-deterministic Semantics.

1 Introduction

Over the last decades logicians like Krister Segerberg [16], David Makinson [11] Heinrich Wansing [17], Lloyd Humberstone [6] started their investigations of modal logics with a very a weak system of modal logic, they either call $\mathbf{L}_0$, **PC** or **S**. [2] Following Humberstone in [6, p. 18] we define the smallest *modal* system as a set **S** in the language of classical propositional logic (**CPL**) containing one additional unary operator, usually $\Box$, where $\Diamond$ is defined as usual, satisfying the following properties:

- **S** contains all (classical) tautologies
- **S** is closed under uniform substitution
- **S** is closed under Modus Ponens

It is obvious from this definition that no specific reference to the primitive modal operator $\Box$ is present, which is usually justified in the sense that one was looking at the *broadest possible definition of what a modal logic might be* [6]. The joint idea of how to proceed with the smallest modal system would then be to extend the axiomatization by certain rules or axioms for the modal

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[2] In the more recent [5] this system is called **0** with the intended reading that there are zero restrictions for the modal operator.

operator and thus gaining a structured way of describing various systems of modal logic.

What is missing in all those representation is a semantics for the smallest modal system that differs from the semantics of **CPL**, by giving some meaning to the modal operator, which is kept for all extensions of the smallest modal system. In what follows, we will provide such a semantics, by building upon the framework of non-deterministic semantics systematically introduced by Arnon Avron and his collaborators, cf. [1], but already used in the context of modal logics by Yuri Ivlev and John Kearns, cf. [7,8], [9,10], and further developed more recently for example in [2,5,12,14,15].

The question whether this semantics has more explanatory power for the notion of what necessity or possibility are supposed to *mean*, compared to possible worlds semantics, neighborhood frames or algebraic approaches, however, will be postponed for another time.

This short paper is then structured as follows, we will give a definition of the semantics for the minimal modal system **M** together with a corresponding sound and complete axiomatization. We will, however, omit due to space restrictions proofs of soundness and completeness. Then we will briefly discuss extensions of **M** by adding both, specific modal axioms and rules and thus presenting a uniform semantics for **M** and its extensions.

2 The minimal modal system M

For the remainder of this paper we assume a propositional language $\mathcal{L}$, consisting of a finite set $\{\neg, \rightarrow, \circ\}$ [3] of propositional connectives and a countable set of propositional parameters. Furthermore, we denote by Form the set of formulas defined as usual in $\mathcal{L}$. We denote formulas of $\mathcal{L}$ by A, B, C, etc. and sets of formulas of $\mathcal{L}$ by Γ, Δ, Σ, etc.

Definition 2.1 An **M**-relational-interpretation is a relation ρ, between formulas and the values 1 and i (i.e., $\rho \subseteq \mathsf{Form} \times \{1, i\}$) such that ρ satisfies the following:

$$\begin{array}{rcl} \neg A\rho 1 & \text{iff} & \text{not } A\rho 1 \\ A{\rightarrow}B\rho 1 & \text{iff} & \text{not } A\rho 1 \text{ or } B\rho 1 \\ \circ A\rho 1 & \text{iff} & A\rho i \end{array}$$

Based on these, A is a *relational* **M**-*consequence* of Γ ($\Gamma \models_{\mathbf{M}} A$) iff for every **M**-relational-interpretation ρ, if $B\rho 1$ for every $B \in \Gamma$ then $A\rho 1$.

Remark 2.2 The first two conditions on ρ reflect the (classical) truth-conditions for the non-modal operators $\neg$ and $\rightarrow$, while the third condition says that any modalized formula A is true, i.e., related to 1, iff A is related to the modal value i. This semantics is not unlike the one for Belnap-Dunn logic, more commonly known as **FDE**, [4] cf. [4], with the main difference, that

[3] We prefer to use $\circ$, rather than $\Box$ or $\Diamond$ because $\circ$ does not have any properties yet and can be interpreted as either.

[4] In fact, one can easily transform this relational semantics in a many-valued one.

nothing is said about formulas standing in relation to i, that is where the non-determinacy comes into play. That is, however, not to say that we want to treat i as a semantic value that is supporting truth or falsity. In our opinion, the value i falls in a different semantic category. A formula related to i has some modal flavor, what this is supposed to be is at this point not determined.

Proof-theoretically we obtain an Hilbert-style calculus for **M** by taking any axiomatization for classical propositional logic.

Definition 2.3 The system **M** consists of the following axioms and a rule of inference.

$$A\to(B\to A) \quad \text{(A1)}$$
$$(A\to(B\to C))\to((A\to B)\to(A\to C)) \quad \text{(A2)}$$
$$(\neg B\to\neg A)\to(A\to B) \quad \text{(A3)}$$
$$\frac{A \quad A\to B}{B} \quad \text{(MP)}$$

Finally, we write $\Gamma \vdash_{\mathbf{M}} A$ if there is a sequence of formulas $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence either (i) belongs to Γ; (ii) is an axiom of **M**; (iii) is obtained by (MP) from formulas preceding it in sequence.

Remark 2.4 It is apparent that **M** satisfies all the properties of the smallest modal system given by Humberstone.

Due to space restrictions we will leave out the proofs of soundness and completeness and only state the results. We will refer the sceptical reader to [12,14] where similar proofs are stated.

Theorem 2.5 (Soundness and Completeness) *For all*

$$\Gamma \cup \{A\} \subseteq \mathsf{Form}\,,\ \Gamma \vdash_{\mathbf{M}} A \textit{ iff } \Gamma \models_{\mathbf{M}} A.$$

3 Extensions of M

In this section we will briefly show that the presented framework is powerful enough to recapture prominent normal and non-normal modal logics. It is important to recognize that some of the results of this section were already presented in previous work, cf. [2,3,5,7,8,12,14,15]. However, the presentation, in particular the relational semantics, is unique to this short paper. Furthermore, the discussion of adding modal rules other than the rule of necessitation is to the best of the authors' knowledge nowhere to be found in previous work.

3.1 Adding axioms to M

We will now list some axioms that can be added, not necessarily at the same time, to the axiomatization of **M** and introduce the corresponding conditions for the relational semantics. For the matter of representation we pair the axioms such that they can be intuitively read as $\Box$- or $\Diamond$-versions of the same axiom.

$\circ A \to A$	$A \to \circ A$
$A\rho i$ only if $A\rho 1$	(not $A\rho i$) only if (not $A\rho 1$)

$\circ A \to \neg \circ \neg A$	$\neg \circ \neg A \to \circ A$
$\neg A\rho i$ only if (not $A\rho i$)	(not $\neg A\rho i$) only if $A\rho i$
$A \to \circ \circ A$	$\circ \circ A \to A$
(not $\circ A\rho i$) only if (not $A\rho 1$)	$\circ A\rho i$ only if $A\rho 1$
$\circ A \to \circ \circ A$	$\circ \circ A \to \circ A$
(not $\circ A\rho i$) only if (not $A\rho i$)	$\circ A\rho i$ only if $A\rho i$
$\circ(A \to B) \to (\circ A \to \circ B)$	$(\circ A \to \circ B) \to \circ(A \to B)$
$A \to B\rho i$ only if (not $A\rho i$) or $B\rho i$	(not $A \to B\rho i$) only if $A\rho i$ and (not $B\rho i$)

Remark 3.1 As the conditions on the relational interpretations ρ show, adding axioms to **M** will lead to a restriction on the relational-interpretation ρ. This restriction might be accomplished by either eliminating some non-determinacy or eliminating possible combinations of 1 and i for the relational-interpretation ρ. In particular, the relational semantics obtained by adding axioms to **M** will be a refinement of the relational semantics for **M**. Cf. [1, Def. 30, p. 16] for the notion of refinement.

The proofs for soundness and completeness for all extensions of **M**, mentioned above, can be similarly obtained as for **M** itself.

3.2 Adding modal rules to M

It was recently shown, cf. [5], that Dugundji's negative result concerning many-valued semantics for modal logics applies to non-deterministic many-valued semantics, or relational non-deterministic semantics for that matter, as well. That is, modal logics with rules concerning the modal operators, can not be expressed solely in terms of the relational non-deterministic semantics presented above. Among all the rules for modal operators some of the most prominent are the following:

$$\frac{A}{\circ A}\ \text{(RN)} \qquad \frac{A \to B}{\circ A \to \circ B}\ \text{(Mon)} \qquad \frac{A \to B}{\circ B \to \circ A}\ \text{(Con)} \qquad \frac{A \leftrightarrow B}{\circ A \leftrightarrow \circ B}\ \text{(Ext)}$$

In order to make use of the relational semantics such that they can express above rules we need to define a hierarchy of relations over the set of all relations and then filter out those relations that are relevant for our purpose. This method was already developed by John Kearns in [9] and further developed and put in to the context of non-deterministic semantics by Omori and Skurt in [12], as well as Coniglio et. al. in [3].

Definition 3.2 Let $\mathbf{M}^+$ be an extension of **M** obtained by adding exactly one of the rules (RN), (Mon), (Con) or (Ext).

We write $\vdash_{\mathbf{M}^+} A$ for, there is a proof for A in $\mathbf{M}^+$ if there is a sequence of formulas $B_1, \ldots, B_n, A$ $(n \geq 0)$, such that every formula in the sequence either (i) is an axiom of $\mathbf{M}^+$; or (ii) is obtained by (MP) or one of modal rules from formulas preceding it in the sequence. Moreover, we define $\Gamma \vdash_{\mathbf{M}^+} A$

iff for a finite subset Γ' of Γ, $\vdash_{\mathbf{M}^+} C_1 \to (C_2 \to (\cdots(C_n \to A)\cdots))$ where $C_i \in \Gamma'(1 \leq i \leq n)$.

On the semantic side, we need to define a hierarchy on the set of relations and consider for evaluating sentences only a certain subset of the set of all relations. Since we want to give the definition of the hierachy of relations for all of the above model rules simultaneously, and because all of the above rules are single premise/conclusion rules we will make use of the following notation $\frac{\mathsf{Premise}}{\mathsf{Conclusion}}$ [5] for the coming definition. E.g., in case of (Mon) we have $\mathsf{Premise} = A \to B$ and $\mathsf{Conclusion} = \circ A \to \circ B$.

Definition 3.3 Let ρ be a relation $\rho \subseteq \mathsf{Form} \times \{1, i\}$. Then,

- ρ is a *0th-level-*$\mathbf{M}^+$*-relation* if ρ is a $\mathbf{M}$-relational-interpretation.
- ρ is an $m+1$*st-level-*$\mathbf{M}^+$*-relation* iff ρ is an mth-level-$\mathbf{M}^+$-relation with $\underline{\mathsf{Conclusion}\rho 1}$ for all sentences $\mathsf{Premise}$, if $\underline{\mathsf{Premise}\rho' 1}$ for any mth-level-$\mathbf{M}^+$-relation ρ'

Based on these, we define ρ to be a $\mathbf{M}^+$*-relation* iff ρ is mth-level-$\mathbf{M}^+$-relation for any $m \geq 0$.

Remark 3.4 Note first that $\mathbf{M}^+$ is obtained by adding exactly one modal rule to $\mathbf{M}$ or its extensions. The addition of more than one modal rule, e.g., (Mon) together with (Con) still needs to be investigated. Secondly, all those considerations apply of course to the extensions of $\mathbf{M}$, discussed in the last section, as well. Lastly, examples of how the hierarchy filters out relations can be found for example in [12], were Def. 3.3 is defined in terms of valuations.

Theorem 3.5 (Soundness and Completeness) *For all*

$$\Gamma \cup \{A\} \subseteq \mathsf{Form}\,,\ \Gamma \vdash_{\mathbf{M}^+} A \text{ iff } \Gamma \models_{\mathbf{M}^+} A.$$

The proofs of soundness and completeness for (RN) have been spelled out at length in [12], while the proofs for the other modal rules have not been published yet.

4 Conclusion

Within the framework of non-deterministic relational semantics we presented a semantics for the smallest modal logic, discussed by Segerberg, Makinson, Wansing or Humberstone. As we showed, this semantics can be extended in a structured way in order order capture other normal and non-normal modal systems, even modal logics with no corresponding Kripke semantics, cf. [13].

One of the advantages of the presented framework for the smallest modal logic is that we do not need to fix our understanding of the modal operator $\circ$, i.e., whether it is meant to be possibility or necessity. This leads to some freedom of adding axioms, modal rules or even more modal operators in order to construct several normal and non-normal modal systems.

[5] Note that Premise and Conclusion are formulas/formula schemes of Form

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A labelled proof system for ignorance

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Abstract

We provide a unified framework expressing three types of ignorance formalized via three primitive modalities: ignorance whether, ignorance as unknown truth, and disbelieving ignorance. To do this, we introduce a class of Kripke models, which interpret these types of ignorance. We then define a labelled sequent calculus and prove its soundness and completeness with respect to these models. Completeness is proved via a countermodel construction from failed proof search.

Keywords: Epistemic logic, Labelled sequent calculus, Ignorance representation.

1 Introduction

Several recent works in epistemic logic focus on finding a way to model the notion of ignorance (see, e.g., [8], [6], [1], [3]). One of the difficulties in achieving this task is that there is no agreement on which notion of ignorance to model. Indeed, van der Hoek & Lomuscio [8] take ignorance to be 'not knowing whether'; Steinsvold [6] considers ignorance as 'unknown truth'; finally, Kubyshkina and Petrolo [3] introduce a primitive ignorance operator relying on the factive nature of ignorance. We argue that these three different approaches should not be considered as exclusive alternatives, but as representing different aspects of the polysemic notion of ignorance, and should coexist in the same formal framework. On the basis of this pluralist view, our main objective is to provide a unified framework expressing all the aforementioned types of ignorance, in order to analyse their behaviour and interactions.

We introduce a class of Kripke models, *ignorance models*, which interpret the three types of ignorance. We then define a labelled sequent calculus called $lab\mathsf{WUDI}$, and prove its soundness and completeness with respect to ignorance models. Completeness is proved by constructing a countermodel from a failed and finite proof search tree. Future work includes the definition of an Hilbert-style axiom system capturing the valid formulas of the models, the introduction of a knowledge operator in ignorance models, and an analysis of the structural properties of $lab\mathsf{WUDI}$, most notably cut-elimination.

2 Ignorance models

Given a countable set of propositional variables $Atm = \{p, q, \dots\}$, formulas of our language are constructed using the following grammar: $\phi ::= p \mid \bot \mid$

$\phi \to \phi \mid \Box\phi \mid I^w\phi \mid I^u\phi \mid I^d\phi$. Negation is defined as $\neg\phi := \phi \to \bot$, and propositional connectives are standardly defined. Operator I^w, for *ignorance whether*, was introduced by van der Hoek & Lomuscio [8]; I^u, for *unknown truth*, by Steinsvold [6,7], and I^d by Kubyshkina & Petrolo [3]. Differently from [3], we interpret I^d as representing *disbelieving ignorance*, which is characterized by Peels [5] as follows: "[a subject] S is disbelievingly ignorant that p iff (i) it is true that p, and (ii) S disbelieves that p." Each ignorance operator has a complete Hilbert-style system, but no unified axiomatisation for the three ignorance operators has been studied.

Definition 2.1 An *ignorance model* is a triple $\mathcal{M} = \langle W, R, v\rangle$, where W is a set of possible worlds, $R \subseteq W \times W$ is a binary relation and $v : Atm \to 2^W$ is a valuation of propositional variables. We assume R to satisfy the *two-worlds property*, that is: for all $x \in W$, there is a $y \in W$ such that xRy and $x \neq y$. The satisfiability relation of formulas in a world x of a model $\mathcal{M}$ is defined as:

- $\mathcal{M}, x \models p$ *iff* $x \in v(p)$ and $\mathcal{M}, x \not\models \bot$;
- $\mathcal{M}, x \models \phi \to \psi$ *iff* $\mathcal{M}, x \not\models \phi$ or $\mathcal{M}, x \models \psi$;
- $\mathcal{M}, x \models \Box\phi$ *iff* for all $y \in W$, if xRy then $\mathcal{M}, y \models \phi$;
- $\mathcal{M}, x \models I^w\phi$ *iff* there is $y \in W$ s.t. xRy and $\mathcal{M}, y \not\models \phi$ and there is $z \in W$ s.t. xRz and $\mathcal{M}, z \models \phi$;
- $\mathcal{M}, x \models I^u\phi$ *iff* $\mathcal{M}, x \models \phi$ and there exists $y \in W$ s.t. xRy and $\mathcal{M}, y \not\models \phi$;
- $\mathcal{M}, x \models I^d\phi$ *iff* $\mathcal{M}, x \models \phi$ and for all $y \in W$, if xRy and $y \neq x$ then $\mathcal{M}, y \not\models \phi$.

We say that ϕ *is valid in* $\mathcal{M}$ and write $\mathcal{M} \models \phi$ if $\mathcal{M}, x \models \phi$ for all x in W. If for all $\mathcal{M}$ we have $\mathcal{M} \models \phi$, we say that ϕ *is valid*, and write $\models \phi$.

Ignorance whether and unknown truth can be defined in terms of the $\Box$ operator as follows: $I^w\phi := \neg\Box\phi \wedge \neg\Box\neg\phi$ and $I^u\phi = \neg\Box\phi \wedge \phi$. Interestingly, disbelieving ignorance is not definable in terms of $\Box$ in none of the standard frames (K, T, S4, and S5), see [2]. Since we here focus on ignorance operator, we take I^w and I^u as primitive in our language.

The two-worlds property ensures that all worlds in a model have access to some world other than themselves. This allows to avoid some counterintuitive consequences: for instance, when evaluating formulas at a one-world model $\mathcal{M}$, we get that $\mathcal{M} \models I^d\top$, $\mathcal{M} \models \neg I^w\top$, and $\mathcal{M} \models \neg I^u\top$. Indeed, it seems implausible that an agent is disbelievingly ignorant of a tautology, but she is not ignorant of its truth (neither in the sense of I^w, nor of I^u). By assuming the two-worlds property we obtain validity of formula $\neg I^d\top$.

3 Labelled sequent calculus

In this section, we shall introduce a labelled calculus *lab*WUDI, following the methodology from [4]. We enrich our language by an infinite set of variables, called *labels*, which we denote $x, y, z \ldots$. Then, *relational atoms* have the form xRy or $x \neq y$, and *labelled formulas* have the form $x : \phi$. A *labelled sequent* has the form $\Gamma \Rightarrow \Delta$, where Γ is a multiset of relational atoms and labelled formulas and Δ is a multiset of labelled formulas.

$$\mathsf{init}\ \frac{}{x:p,\Gamma\Rightarrow\Delta,x:p}\qquad \bot\ \frac{}{x:\bot,\Gamma\Rightarrow\Delta}\qquad \mathsf{2w}\ \frac{xRy,x\neq y,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}\ *$$

$$\to_{\mathsf{L}}\ \frac{\Gamma\Rightarrow\Delta,x:\phi\quad x:\psi,\Gamma\Rightarrow\Delta}{x:\phi\to\psi,\Gamma\Rightarrow\Delta}\qquad \to_{\mathsf{R}}\ \frac{x:\phi,\Gamma\Rightarrow\Delta,x:\psi}{\Gamma\Rightarrow\Delta,x:\phi\to\psi}$$

$$\Box_{\mathsf{L}}\ \frac{xRy,x:\Box\phi,y:\phi,\Gamma\Rightarrow\Delta}{xRy,x:\Box\phi,\Gamma\Rightarrow\Delta}\qquad \Box_{\mathsf{R}}\ \frac{xRy,\Gamma\Rightarrow\Delta,y:\phi}{\Gamma\Rightarrow\Delta,x:\Box\phi}\ *\qquad I^d_{\mathsf{L1}}\ \frac{x:I^d\phi,x:\phi,\Gamma\Rightarrow\Delta}{x:I^d\phi,\Gamma\Rightarrow\Delta}$$

$$I^d_{\mathsf{L2}}\ \frac{xRy,x\neq y,x:I^d\phi,\Gamma\Rightarrow\Delta,y:\phi}{xRy,x\neq y,x:I^d\phi,\Gamma\Rightarrow\Delta}\qquad I^d_{\mathsf{R}}\ \frac{\Gamma\Rightarrow\Delta,x:\phi\quad xRy,x\neq y,y:\phi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,x:I^d\phi}\ *$$

$$I^u_{\mathsf{L}}\ \frac{xRy,x:\phi,\Gamma\Rightarrow\Delta,y:\phi}{x:I^u\phi,\Gamma\Rightarrow\Delta}\ *\qquad I^u_{\mathsf{R}}\ \frac{\Gamma\Rightarrow\Delta,x:\phi\quad x:\Box\phi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,x:I^u\phi}$$

$$I^w_{\mathsf{L}}\ \frac{xRy,xRz,y:\phi,\Gamma\Rightarrow\Delta,z:\phi}{x:I^w\phi,\Gamma\Rightarrow\Delta}\ *\qquad I^w_{\mathsf{R}}\ \frac{x:\Box\neg\phi,\Gamma\Rightarrow\Delta\quad x:\Box\phi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,x:I^w\phi}$$

$*$: y is fresh, i.e., it does not occur in $\Gamma\cup\Delta$.

Fig. 1. Sequent calculus $lab\mathsf{WUDI}$

The rules of $lab\mathsf{WUDI}$ are illustrated in Figure 1. The calculus features only one structural rule, 2w, expressing the two-worlds property. Propositional rules and the rules for $\Box$ are standard. The rules for ignorance operators have been defined based on the truth condition of the operators at ignorance models. The condition for I^d on the left is captured by a pair of rules, one of which only introducing formulas within the label of the principal formula, x. Rules I^u_{R} and I^w_{R} introduce $\Box$-formulas in its premisses, needed to express the universal conditions in the negated truth conditions for I^u and I^w respectively.

Labelled sequents do not have a direct formula interpretation, and thus we need to interpret them over ignorance models to prove soundness of the calculus, which is straightforward.

Definition 3.1 Given a labelled sequent $\Gamma\Rightarrow\Delta$ and an ignorance model $\mathcal{M}=\langle W,R,v\rangle$, let $S=\{x\mid x\in\Gamma\cup\Delta\}$ and $\rho:S\to W$. We define the following relation: $\mathcal{M},\rho\models xRy$ iff $\rho(x)R\rho(y)$; $\mathcal{M},\rho\models x\neq y$ iff $\rho(x)\neq\rho(y)$; and $\mathcal{M},\rho\models x:\phi$ iff $\rho(x)\models\phi$.

$\Gamma\Rightarrow\Delta$ *is satisfied at* $\mathcal{M}$ *under* ρ if, if for all formulas $\phi\in\Gamma$ it holds that $\mathcal{M},\rho\models\phi$, then there exists a formula $\psi\in\Delta$ such that $\mathcal{M},\rho\models\psi$. Then, $\Gamma\Rightarrow\Delta$ *is valid in* $\mathcal{M}$ if the sequent is satisfied at $\mathcal{M}$ for all ρ. Finally, $\Gamma\Rightarrow\Delta$ is valid if the sequent is valid in all models.

Theorem 3.2 (Soundness) *If there is a derivation of* $\Rightarrow x:\phi$, ϕ *is valid.*

To prove completeness of $lab\mathsf{WUDI}$ with respect to ignorance models, we first need to show termination of the calculus. The rule 2w, if applied in an unrestricted way to a sequent, leads to non-termination of root-first proof search. Thus, we shall introduce a *proof search strategy*, allowing one to always obtain finite proof search trees for a given sequent, and then we show how to extract a finite countermodel from a branch of a failed proof search tree. We start by defining a measure for labelled formulas.

Definition 3.3 We define the *weight of a labelled formula* as follows: $w(xRy) = w(x \neq y) = 0$ and, for a labelled formula $x : \phi$, we set $w(x : \chi) = w(\chi)$, where $w(\chi)$ is inductively defined as follows: $w(p) = w(\perp) = 1$; $w(\phi \to \psi) = w(\phi) + w(\psi) + 1$; $w(K\phi) = w(\phi) + 2$; and $w(I^d\phi) = w(I^u\phi) = w(I^w\phi) = w(\phi) + 3$.

To define a proof search strategy, we first need to define the notion of *saturated sequent.* Intuitively, a sequent is saturated if it is not an initial sequent and if all the rules have been non-redundantly applied to it. More formally, given a branch $\mathcal{B} = \{\Gamma_i \Rightarrow \Delta_i\}_{i>0}$ in a proof search tree and a sequent $\Gamma_n \Rightarrow \Delta_n$ in $\mathcal{B}$, let $\downarrow \Gamma_n = \bigcup_{i=1}^{n} \Gamma_i$ and $\downarrow \Delta_n = \bigcup_{i=1}^{n} \Delta_i$. Moreover, given two labels z and x occurring in a sequent $\Gamma_n \Rightarrow \Delta_n$, we write $\mathsf{For}(z) = \mathsf{For}(x)$ meaning that the set of formulas labelled by z and occurring in $\downarrow \Gamma_n$ coincides with the set of formulas labelled with x and occurring in $\downarrow \Gamma_n$, and similarly for $\downarrow \Delta_n$. We show the condition for $\mathsf{2w}$ and the standard condition for I^d_{R}:

(I^d_{R}) If $x : I^d\phi$ is in $\downarrow \Delta_n$, then either $x : \phi$ is in $\downarrow \Delta_n$ or for some y, xRy, $x \neq y$ and $y : \phi$ are in $\downarrow \Gamma_n$.

($\mathsf{2w}$) For all x in $\downarrow \Gamma_n \cup \downarrow \Delta_n$, it holds that either xRy and $x \neq y$ are in Γ for some y, or that zRx and $z \neq x$ are in Γ, for some z such that $\mathsf{For}(z) = \mathsf{For}(x)$.

A labelled sequent is *saturated* if it meets the saturation conditions for all the rules, and if it not an instance of $\perp$ or init.

Next, we define our *proof search strategy* as follows: given a sequent, we first apply to it rules that do not introduce bottom-up new labels (*static rules*), and rules that do introduce new labels (*dynamic rules*), except for $\mathsf{2w}$. Once all the other rules have been applied, we apply $\mathsf{2w}$, taking care of not applying the rule to a label x if one of the two conditions described in the saturation condition is met. Then, we prove the following:

Theorem 3.4 (Termination) *Root-first proof search for a sequent $\Rightarrow x : \phi$ built in accordance with the strategy comes to an end in a finite number of steps, and each leaf of the proof-search tree contains either an initial sequent or a saturated sequent.*

Proof. The only source of non-termination in a branch $\mathcal{B}$ is rule $\mathsf{2w}$. We associate to $\mathcal{B}$ a tree whose nodes are the labels occurring in the branch and such that there is an edge between two nodes x and y only if both xRy and $x \neq y$ occur in $\downarrow \Gamma_n$. Then, we prove that the tree is finite, by showing that every node has a finite number of children (immediate) and that every chain in the tree has finite length (also immediate, relying on the second disjunct of the saturation condition for $\mathsf{2w}$). Thus, there are only finitely many labelled formulas which can non-redundantly occur in a branch. $\square$

To conclude, we sketch the proof of completeness:

Theorem 3.5 (Completeness) *If ϕ is valid, there is a derivation of $\Rightarrow x : \phi$.*

Proof. We prove the counterpositive. Suppose that $\Rightarrow x : \phi$ is *not* derivable in labWUDI. By termination, if ϕ is not derivable then there is a proof search branch $\mathcal{B}$ whose upper node is occupied by a saturated sequent, $\Gamma_n \Rightarrow \Delta_n$. We construct a model $\mathcal{M}^{\mathcal{B}}$ that satisfies all formulas in $\downarrow \Gamma_n$ and falsifies all formulas in $\downarrow \Delta_n$. As a consequence, $\mathcal{M}^{\mathcal{B}}$ falsifies $x : \phi$ and, by definition, there exists a world and a model that falsifies ϕ. Thus ϕ is not valid.

We construct a model $\mathcal{M}^{\mathcal{B}} = \langle W^{\mathcal{B}}, R^{\mathcal{B}}, v^{\mathcal{B}} \rangle$ from the branch $\mathcal{B}$ as follows: $W^{\mathcal{B}} = \{x \mid x \in \downarrow \Gamma_n \cup \downarrow \Delta_n\}$, $R^{\mathcal{B}} = \{(x, y) \mid xRy \in \Gamma\}$ and $v^{\mathcal{B}}(p) = \{x \in W^{\mathcal{B}} \mid x : p \in \Gamma\}$. Distinct variables in $\downarrow \Gamma_n \cup \downarrow \Delta_n$ get mapped to distinct elements in $W^{\mathcal{B}}$. As a consequence, whenever $x \neq y$ occurs in Γ, the two variables x and y get mapped to two distinct elements in $W^{\mathcal{B}}$.

As it is, $\mathcal{M}^{\mathcal{B}}$ does not satisfy the two-worlds condition. We modify it as follows. Whenever we have a world x that has no access to worlds other than itself, by the saturation condition (2w) there needs to be a world z such that zRx and $z \neq x$ occur in Γ. We add $(x, z) \in R^{\mathcal{B}}$. Since $x \neq z$ occur in Γ, we conclude that x satisfies the two-worlds condition.

To conclude the proof, we set ρ such that $\rho(x) = x$, for all $x \in \downarrow \Gamma_n \cup \downarrow \Delta_n$, and we show by induction on the weight of formulas that the following hold, for any labelled formula $\mathcal{F}$: *a)* If $\mathcal{F} \in \downarrow \Gamma_n$, then $\mathcal{M}^{\mathcal{B}}, \rho \models \mathcal{F}$ and *b)* If $\mathcal{F} \in \downarrow \Delta_n$, then $\mathcal{M}^{\mathcal{B}}, \rho \not\models \mathcal{F}$. We prove the crucial case in which $x : I^d\phi$ is in $\downarrow \Gamma_n$. By saturation condition (I^d_{L1}) it holds that $x : \phi$ occurs in $\downarrow \Gamma_n$, and by IH $\mathcal{M}^{\mathcal{B}}, \rho \models x : \phi$. Next, take an arbitrary $y \in W^{\mathcal{B}}$ such that xRy and $x \neq y$. If both xRy and $x \neq y$ occur in Γ, we can conclude that $\mathcal{M}^{\mathcal{B}}, \rho \models I^d\phi$ using the saturation condition for (I^d_{L2}) and the IH. Otherwise, xRy and $x \neq y$ have been introduced to meet the two-worlds condition. But this cannot be the case: by definition, y needs to be such that yRx and $y \neq x$ occur in $\downarrow \Gamma_n$, and $\mathsf{For}(y) = \mathsf{For}(x)$. Then, $y : I^d\phi$ is in $\downarrow \Gamma_n$. By saturation condition (I^d_{L1}) $y : \phi$ is in $\downarrow \Gamma_n$, and by saturation condition (I^d_{L2}) $x : \phi$ is in $\downarrow \Delta_n$. Since $x : \phi$ occurs in $\downarrow \Gamma_n$ (see above), $\Gamma_n \Rightarrow \Delta_n$ is an instance of init, because labelled compound formulas get decomposed in their simpler components, and labelled atomic formulas never disappear bottom-up. Thus, we get a contradiction with the assumption that $\Gamma_n \Rightarrow \Delta_n$ is saturated. $\square$

Example 3.6 A failed derivation of sequent $\Rightarrow x : I^d p \to \bot$, from which we construct the model $\mathcal{M} = \langle W, R, v \rangle$, descrided below. It holds that $\mathcal{M}, x \not\models I^d p \to \bot$, and that $\mathcal{M}$ satisfies the two worlds property.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRy, x \neq y, yRz, y \neq z, zRk, z \neq k, x : p, x : I^d p \Rightarrow x : \bot, y : p}\,\mathsf{fail}}{xRy, x \neq y, yRz, y \neq z, x : p, x : I^d p \Rightarrow x : \bot, y : p}\,\mathsf{2w}}{xRy, x \neq y, x : p, x : I^d p \Rightarrow x : \bot, y : p}\,\mathsf{2w}}{xRy, x \neq y, x : p, x : I^d p \Rightarrow x : \bot}\,\to_{\mathsf{L2}}}{x : p, x : I^d p \Rightarrow x : \bot}\,\mathsf{2w}}{x : I^d p \Rightarrow x : \bot}\,I^d_{\mathsf{L1}}}{\Rightarrow x : I^d p \to \bot}\,\to_{\mathsf{R}}
$$

$$W = \{x, y, z, k\}; \quad R = \{(x, y), (y, z), (z, k), (k, z)\}; \quad v(p) = \{x\}.$$

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A Verified Proof of Craig Interpolation for Basic Modal Logic via Tableaux in Lean

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Abstract

We present a formalisation of tableaux for Basic Modal Logic (K) in the Lean proof assistant. We cover soundness, completeness, and Craig Interpolation via tableaux. In the future we aim to extend this proof from Basic Modal Logic to Propositional Dynamic Logic (PDL). Specifically, this is a first step towards a formal verification of Borzechowski (1988) which claims that PDL has Craig Interpolation.

Keywords: Modal Logic, Interpolation, Tableaux, Interactive Theorem Proving

1 Introduction

It is well-known that many modal logics have the Craig interpolation property (see Theorem 3.1 below). However, for Propositional Dynamic Logic (PDL) [4] this is considered an open question. Recently the proof attempt by [2] was translated from German to English [3]. As a first step towards a formal verification of this proof, here we formalise the Basic Modal Logic analogue of the PDL tableaux in [2] in the proof assistant *Lean*. We then show soundness, completeness and Craig interpolation via tableaux. The full code is publicly available at `https://github.com/m4lvin/tablean`.

Lean. Lean can be considered both a programming language, a theorem prover and a proof assistant. It is based on Type Theory; its main developer is L. de Moura [9]. A large amount of mathematics has been formalised in Lean and is maintained by the *mathlib* project [12]. For this project we use the extended community version of Lean 3.42.1 and refer to `https://leanprover-community.github.io` for further details.

Related Work. The mathlib project does not contain any modal logic so far, but several formalisation projects of modal logics have been done in Lean. Most similar to ours is the work by Wu and Goré [13], who implement a verified decision procedure based on tableaux for the basic modal logics K, KT and S4. They do not discuss interpolation. Other related Lean projects (without

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tableaux) include the completeness of S5 [1], and formalisations of Public Announcement Logic [7,10]. Modal logics have also been formalised in other proof assistants, e.g. Epistemic Logic in Isabelle [5]. Another similar Isabelle project is a formalised proof of interpolation for classical propositional logic [8].

Basic Definitions. We consider basic modal logic, given by the BNF syntax $\varphi ::= \bot \mid p \mid p\neg\varphi \mid \varphi\wedge\varphi \mid \Box\varphi$ where $p \in \mathsf{P}$ for some set P of atomic propositions. We use the abbreviations $\varphi \to \psi := \neg(\varphi\wedge\neg\psi)$ and $\varphi\vee\psi := \neg(\neg\varphi\wedge\neg\psi)$. For any φ, we denote by $\mathsf{voc}(\varphi)$ its *vocabulary*. E.g., $\mathsf{voc}((p\vee q)\wedge(\Box q \to \diamond r)) = \{p, q, r\}$.

We use standard Kripke models $\mathcal{M} = (W, R, V)$ where $R \subseteq W \times W$ and $V : W \to \mathsf{Prop}$. As our first example of Lean code, the following defines the standard semantics. Here `formula` is an inductive type for the grammar above, and `char` plays the role of the set P for atomic propositions.

```
structure kripkeModel (W : Type) : Type :=
  (val : W → char → Prop)
  (rel : W → W → Prop)

def evaluate {W : Type} : (kripkeModel W × W) → formula → Prop
| (M,w) ⊥         := false
| (M,w) (· p)     := M.val w p
| (M,w) (~ φ)     := ¬ evaluate (M,w) φ
| (M,w) (φ ⅄ ψ)   := evaluate (M,w) φ ∧ evaluate (M,w) ψ
| (M,w) (□ φ)     := ∀ v : W, (M.rel w v → evaluate (M,v) φ)
```

2 Tableaux

The tableaux system we use consists of the following five *local rules* plus one modal rule. Here X is a set of formulas and $\varnothing$ indicates a *closed* branch.

$$(\bot)\,\frac{X,\bot}{\varnothing} \qquad (\neg)\,\frac{X,\varphi,\neg\varphi}{\varnothing} \qquad (\neg\neg)\,\frac{X,\neg\neg\varphi}{X,\varphi} \qquad (\wedge)\,\frac{X,\varphi\wedge\psi}{X,\varphi,\psi} \qquad (\neg\wedge)\,\frac{X,\neg(\varphi\wedge\psi)}{X,\neg\varphi \mid X,\neg\psi}$$

The five local rules are represented in Lean as follows.

```
inductive localRule : finset formula → finset (finset formula) → Type
| bot {X    } (h : ⊥          ∈ X) : localRule X ∅
| not {X φ  } (h : φ ∈ X ∧ ~φ ∈ X) : localRule X ∅
| neg {X φ  } (h : ~~φ        ∈ X) : localRule X { X \ {~~φ}     ∪ {φ}   }
| con {X φ ψ} (h : φ ⅄ ψ      ∈ X) : localRule X { X \ {φ⅄ψ}     ∪ {φ,ψ} }
| nCo {X φ ψ} (h : ~(φ ⅄ ψ)   ∈ X) : localRule X { X \ {~(φ⅄ψ)} ∪ {~φ}
                                                , X \ {~(φ⅄ψ)} ∪ {~ψ}   }
```

The *modal rule* uses the *projection* $X_\Box := \{\varphi \mid \Box\varphi \in X\}$. It may only be applied when X is *simple*, i.e. consists of (negated) atoms and boxed formulas.

$$(\neg\Box)\,\frac{X,\neg\Box\varphi}{X_\Box,\neg\varphi}$$

Example 2.1 Fig. 1 shows a tableau for $(\Box\Box(p\wedge q)\vee\Box\bot) \to (\Box(r\vee\Box q)\vee\Box r)$. It consists of four local tableaux, connected by three applications of the modal rule, marked with dotted lines. We spell out all abbreviations using $\neg$ and $\wedge$.

In Lean we define tableaux as two inductive types. For a `localTableau` for X we need either a local rule applicable to X and local tableaux for the `next` nodes, or that X is `simple`. A `closedTableau` for X consists either of a local tableau together with closed tableaux for all its (simple) end nodes, or an application of the modal rule and a closed tableau for the result.

```
inductive localTableau : finset formula → Type
| byLocalRule {X B}
    (_ : localRule X B) (next : Π Y ∈ B, localTableau Y) : localTableau X
| sim {X} : simple X → localTableau X

inductive closedTableau : finset formula → Type
| loc {X} (lt : localTableau X) :
    (Π Y ∈ endNodesOf ⟨X, lt⟩, closedTableau Y) → closedTableau X
| atm {X φ} : ~□φ ∈ X → simple X →
    closedTableau (projection X ∪ {~φ}) → closedTableau X
```

Theorem 2.2 (Soundness and Completeness) *Let X be a finite set of formulas. There is a closed tableau for X if and only if X is unsatisfiable.*

For soundness it suffices to show that all rules preserve satisfiability. For the completeness proof (by contraposition) we use open tableaux (i.e. not ending in $\varnothing$) to build a model satisfying X. Note that having one open tableau does not imply that there is no closed tableau, because the modal rule allows a choice between different formulas. For example, in Fig. 1 in the left branch the first application of $\neg\Box$ on the formula $\neg\Box r$ would lead to an open tableau.

The formal completeness proof in Lean is by induction on the size of X. A crucial part of the completeness proof is Lemma 1 from [2]: a simple set is satisfiable iff it is not closed and all its projections are satisfiable.

```
lemma almostCompleteness :
  Π n X, lengthOfSet X = n → consistent X → satisfiable X

lemma Lemma1_simple_sat_iff_all_projections_sat {X : finset formula} :
  simple X → ( satisfiable X ↔
      (¬ closed X ∧ ∀ R, (~□R) ∈ X → satisfiable (projection X ∪ {~R})) )
```

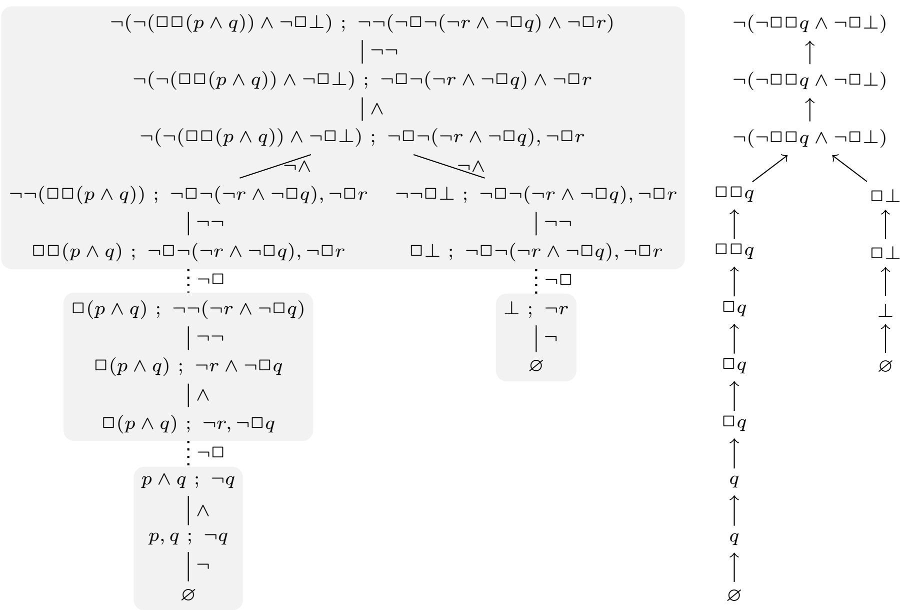

Fig. 1. Example tableau (left) and corresponding tree of interpolants (right).

3 Interpolation via Tableaux

In [11] it was shown that tableaux systems such as the one defined in the previous section can be used to show the following (see also [6, Section 3.8]).

Theorem 3.1 (Craig Interpolation) *For any formulas φ and ψ, if $\varphi \to \psi$ is a tautology, then there exists a formula θ (called* interpolant*) such that $\varphi \to \theta$ and $\theta \to \psi$ are tautologies and $\mathsf{voc}(\theta) \subseteq \mathsf{voc}(\varphi) \cap \mathsf{voc}(\psi)$.*

To show interpolation via tableaux we also need interpolants for partitions. A *partition* of X is a pair of sets of formulas (X_1, X_2) such that $X = X_1 \cup X_2$. We also write $X_1; X_2$ for (X_1, X_2), as done already in Fig. 1.

Definition 3.2 A formula θ is an *interpolant for a partition* $X_1; X_2$ iff $\mathsf{voc}(\theta) \subseteq \mathsf{voc}(X_1) \cap \mathsf{voc}(X_2)$ and both $X_1 \cup \{\neg\theta\}$ and $X_2 \cup \{\theta\}$ are not satisfiable.

```
def partInterpolant (X1 X2 : finset formula) (θ : formula) :=
  voc θ ⊆ (voc X1 ∩ voc X2)  ∧  ¬ satisfiable ( X1 ∪ {~θ} )
                             ∧  ¬ satisfiable ( X2 ∪ { θ} )
```

Lemma 3.3 *A formula $\varphi \to \psi$ is a tautology iff $\{\varphi, \neg\psi\}$ is not satisfiable. Moreover, θ is an interpolant for $\varphi \to \psi$ iff θ is an interpolant for $(\{\varphi\}, \{\neg\psi\})$.*

To prove Theorem 3.1 we first use completeness to get a closed tableau for $\varphi; \neg\psi$ and then use it to define interpolants. While a tableau is built starting at the root, the corresponding tree of interpolants is generated starting at the leaves. Fig. 1 shows how we obtain the interpolant $\Box\Box q \vee \Box\bot$ for Example 2.1.

How interpolants from child nodes are combined to define interpolants for their parent nodes depends on the rule, and the side(s) of the partition where it is applied. A large part of our Lean formalisation is about the choice of interpolants, and showing correctness of the choice. For example, in Fig. 1 we apply $(\neg\wedge)$ on the left side of the partition (`X1`) and take the disjunction of the old interpolants θ_a and θ_b to define the new one. This uses the following.

```
lemma nCoInterpolantX1 {X1 X2 φ ψ θa θb} :
  ~(φ⋏ψ) ∈ X1 →
    partInterpolant (X1 \ {~(φ⋏ψ)} ∪ {~φ}) (X2 \ {~(φ⋏ψ)}) θa →
      partInterpolant (X1 \ {~(φ⋏ψ)} ∪ {~ψ}) (X2 \ {~(φ⋏ψ)}) θb →
        partInterpolant X1 X2 (~(~θa ⋏ ~θb))
```

Finally, the proof that interpolants as in Definition 3.2 always exist is by a structural induction on the `closedTableaux`, and a second induction on the size of the set of formulas for local tableaux.

4 Future Work

Refactoring and optimization. Our project currently has around 2900 lines of code, including around 1700 unique lines. Checking the proof takes several minutes on an Intel i7 CPU with 4.7 GHz. To improve this we plan to avoid general proof tactics such as `finish` and duplicate code.

Towards PDL. Our formalisation is unnecessarily complicated for basic modal logic. For example, separating `closedTableau` and `localTableau` is not needed. But, as mentioned above, our long term goal is to move from BML to PDL and formally verify the proof attempt from [2,3]. Hence we use the same definitions, ideas and high-level proof structure. [2] The next intermediate steps will be multi-modal logic and star-free PDL. Another shortcut could be to leave out tests, as PDL has interpolation if test-free PDL has interpolation [6, Theorem 10.6.2], unless tests are needed for the construction given in [2].

Finally, we did not have space here for a comparison to [13], but we also aim to study whether interpolation could be shown formally in their framework.

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[2] An exception is that [2] uses model graphs to explicitly build counter models from open tableaux. Our formalisation defines model graphs, but they are not yet used.

Linking sanctions to norms in practice (short version)

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Abstract

Within social simulation, we often want agents to interact both with larger systems of norms, as well as respond to their own and other agents norm violations. However, there are no norm specifications that allow us to specify all of these components. To address this issue, this paper introduces the concept of violation *modalities* in CTL^*. These modalities do not only allow us to keep track of violations, but also allow us to define the usual deontic operators, and link norms together.

Keywords: Norms, Sanctions, Norm violations, Deadlines, Repeated obligations.

1 Introduction

For a long time, deontic logics have been used as the basis for normative specification in Computer Science [5,11,10], which means that there are many different formalizations to pick from. However, due to the complex nature of norms, these formalizations are not always applicable in every subfield of computer science. For example, in social simulation, which deals with normative reasoning quite often, these formalizations are hardly used, because they do not give the right kind of handles for the designer of a simulation to work with. In social simulation agents often need to react to the behaviour of other agents, while most of the existing formalizations are more focused on how an agent can select the best action under the influence of a norm. This paper hopes to address this issue by introducing a logic that explicitly represents not only obligations and prohibitions generated from norms, but also represents the norm violations.

We can also find this idea of tracking violations in various reductions in dynamic and temporal logics [5,4]. However, in earlier versions a world was either in violation or not, which made it impossible to have repeating norms or repeating violations.

Other approaches that have tried to solve this problem have done this through the use of *norm instances* [2,7,10], which link the parameters of the

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norm also onto the violation and the sanctions. This allows them to handle norm change [7] as well as planning and norm life cycle management [10]. However, due to the use of these different instances, it is not trivial to see how different norms can interact, or how multiple violations of the same norm can be aggregated. We hope to address this issue by making the violations unique, by using a violation modality.

The new addition to this logic will be the addition of norm violation conditions, deadlines, and repair and punish conditions into the logic. This will allow us to define multiple violation modalities, for both acting and omitting to act, as well as modalities for repaired and punished violations. All of these modalities can occur multiple times on a run without causing problems, and will allow us to link norms to violations, and link norms to other norms.

2 $\mathsf{CTL}^*_{lp}$with extensions

In order to keep track of these violations we will need to extend CTL^*. We need to reason about the past, which means we will need a logic with past operators as well, such as CTL^*_{lp} [3]. To make the violations unique, we also need some kind of clock to keep track of when a violation happened for which we will use the freeze operator $t.\varphi(t)$ from *Timed Propositional Temporal Logic* (TPTL) [1]. We will also need to be able to count how often a violation occurred, for which we will be using some of the definitions from [9]. Lastly, as in [4], we want some notion of causality or responsibility. For this, we will be incorporating elements from [6] to bring in a *seeing to it that* $E_a\varphi$ (STIT) operator.

Definition 2.1 Given a set of propositional variables $p \in \mathcal{P}$, a set of agents $a \in \mathcal{A}$, a set of norm labels $i \in \mathcal{I}$, a set of temporal variables $t \in \mathcal{T}$, and a temporal direction label $d \in \{+, -\}$, the language $\mathcal{L}$ is defined as:

$$
\begin{aligned}
\pi &:= t \mid t + c \mid t - c \\
\varphi &:= p \mid \pi_1 < \pi_2 \mid \pi_1 = \pi_2 \mid \neg\varphi \mid \varphi \wedge \varphi \mid E_a\varphi \mid \mathsf{E}\alpha \mid \mathsf{A}\alpha \mid t.\varphi(t) \\
&\qquad V_{i,a} \mid D_{i,a} \mid R_{i,a} \mid P_{i,a} \\
\alpha &:= \varphi \mid \neg\alpha \mid \alpha \wedge \alpha \mid \mathsf{X}^d\alpha \mid \mathsf{F}^d\alpha \mid \mathsf{G}^d\alpha \mid \alpha\mathsf{U}^d\alpha
\end{aligned}
$$

We will be using the usual abbreviations for disjunction and implication, as well as abbreviations for the other binary operators for comparing times. $V_{i,a}$, $D_{i,a}$, $R_{i,a}$, and $D_{i,a}$ are the new violation/deadline/reparation/punished propositions which we will use to define the violation modalities.

Definition 2.2 A model is a tuple $M = \langle \mathcal{W}, w, \mathcal{R}, T, \pi, V, D, R, P \rangle$ where:

- $\langle \mathcal{W}, \mathcal{R} \rangle$ is a tree of world states with w as its root;
- $T : \mathcal{R} \to 2^{\mathcal{A}}$ is a set of agent labels on elements of $\mathcal{R}$, to signify which agents have control over the transition;
- $\pi : \mathcal{W} \times 2^{\mathcal{P}}$ is a valuation function over the states;
- $V : \mathcal{I} \times \mathcal{W} \times \mathcal{A} \to 2^{\mathcal{W}}$ marks where an agent violated a norm;
- $D : \mathcal{I} \times \mathcal{W} \times \mathcal{A} \to 2^{\mathcal{W}}$ marks where an agent has a deadline for a norm;

- $R : \mathcal{I} \times \mathcal{W} \times \mathcal{A} \to 2^{\mathcal{W}}$ marks where an agent has repaired a norm;
- $P : \mathcal{I} \times \mathcal{W} \times \mathcal{A} \to 2^{\mathcal{W}}$ marks where an agent is punished for a violation.

The depth of a state in the tree is denoted with $depth(\mathcal{W}, \mathcal{R}, s)$, with $depth(\mathcal{W}, \mathcal{R}, w) = 0$. Path formula are evaluated on possibly infinite paths. However, unlike traditional CTL^* approaches, these paths are always coming from the root of the tree. We write $paths(\mathcal{W}, \mathcal{R}, s)$ for all paths going through the state s. If σ is a path, then the j-th state on that path is $\sigma(j)$. $\sigma\restriction_i$ is the finite path consisting of all elements up and to including i.

Definition 2.3 The semantics of the temporal/boolean operators is as in [3,1].

$$\begin{aligned} M, s \models_\tau E_a\varphi \iff & \text{for all } s\mathcal{R}s', M, s' \models_\tau \varphi, \text{ and } a \in T((s, s')), \\ & \text{and } M, w \models_\tau \neg\mathsf{AG}^+\varphi \end{aligned}$$

$$\begin{aligned} M, s \models_\tau V_{i,a} &\iff s \in V(i, a) \qquad & M, s \models_\tau D_{i,a} &\iff s \in D(i, a) \\ M, s \models_\tau R_{i,a} &\iff s \in R(i, a) \qquad & M, s \models_\tau P_{i,a} &\iff s \in P(i, a) \end{aligned}$$

Now, the last thing we need is to be able to count occurrences of a formula on a path. We will adapt the definition from [9] to slice the path itself.

Definition 2.4 For every path σ, the number of states satisfying a path formula α between depth i and j is denoted as $|\sigma|_\alpha^{i:j}$ and defined as $|\{k \mid i \leq k \leq j \text{ and } M, \sigma\restriction_j, k \models_\tau \alpha\}|$.

3 Violation as a modality

With all of this machinery in place, we can now extend our language $\mathcal{L}$ with the violation modalities, written as $\mathrm{Viol}^a_{i,a,t_b,t_v}(\varphi)$. This modality is supposed to be read as "agent a has violated norm i at time t_v by seeing to it that φ, which counts as a violation since t_b". Because we are working with a STIT logic however, there is a difference between acting and failing to act, so we will need two operators, Viol^a and Viol^o. The intuition behind these operators is as follows. For an agent to be in violation for (not) doing φ at time point t_v:

(i) that agent must be in violation of the norm at that time point ($V_{i,a}$ must hold at t_v, and for omission, also $D_{i,a}$);

(ii) the agent must (not) have done φ. In the case of not doing φ, it cannot been done since t_b;

(iii) (not) doing φ must be linked to the world being in violation (φ implies violation for acting, violation implying $\neg\varphi$ for not acting);

(iv) for omissions to act, the agent must not have done φ (or gotten a violation for not doing φ) more often than the deadline passed.

Definition 3.1 Violation modalities are defined as:

$$M, s \models_\tau \mathrm{Viol}^a_{i,a,t_b,t_v}(\varphi) \iff$$
$$(i\&ii)\ M, s \models_\tau (t.(t_v = t \land V_{i,a} \land \mathsf{AX}^-(E_a\varphi)) \lor \mathsf{AF}^-(t.(t_v = t) \land V_{i,a} \land \mathsf{AX}^-(E_a\varphi))$$

(iii) and $M, s \models_\tau \mathsf{AG}^{-}t.((t \geq t_b \wedge t \leq t_v) \rightarrow (\varphi \rightarrow V_{i,a}))$
$M, s \models_\tau \mathrm{Viol}^{o}_{i,a,t_b,t_v}(\varphi) \iff$
(i) $M, s \models_\tau (t.(t_v = t \wedge V_{i,a} \wedge D_{i,a}) \vee \mathsf{AF}^{-}(t.(t_v = t) \wedge V_{i,a} \wedge D_{i,a}),$
(ii) $M, s \models_\tau \mathsf{A}(\neg E_a\varphi \wedge \neg D_{i,a})\mathsf{U}^{-}(D_{i,a} \vee t.(t_b = t))$
(iii) $M, s \models_\tau \mathsf{AG}^{-}t.((t \geq t_b \wedge t \leq t_v) \rightarrow (V_{i,a} \rightarrow \neg\varphi)),$
(iv) and for all $\sigma \in \mathit{paths}(\mathcal{W}, \mathcal{R}, s), |\sigma|^{t_b:t_v}_{D_{i,a}} > |\sigma|^{t_b:t_v}_{t.(\mathrm{Viol}^{o}_{i,a,t_b,t}(\varphi)) \vee E_a\varphi}$

In order to specify the deactivation conditions for reparation, punishment and meta-norms, we also need to know whether or not a violation has yet been repaired/punished for. The intuition behind this is as follows:

(i) there was a violation in the past;
(ii) the agent saw to it that the violation was repaired/they got punished;
(iii) this happened more often than there were unrepaired/punished violations.

Definition 3.2 Repaired norm violation modality is defined as:

$\mathrm{Viol}_{i,a,t_b,t_v}(\varphi) := \mathrm{Viol}^{a}_{i,a,t_b,t_v}(\varphi) \vee \mathrm{Viol}^{o}_{i,a,t_b,t_v}(\varphi)$
$M, s \models_\tau \mathrm{RViol}_{i,a,t_b,t_v,t_r}(\varphi) \iff$
(i) $M, s \models_\tau \mathrm{Viol}_{i,a,t_b,t_v}(\varphi)$
(ii) and $M, s \models_\tau \mathsf{AF}^{-}(t.(t_r = t) \wedge E_a R_{i,a})$
(iii) and for all $\sigma \in \mathit{paths}(\mathcal{W}, \mathcal{R}, s), |\sigma|^{t_v:t_r}_{E_a R_{i,a}} \geq |\sigma|^{t_b:t_v}_{t.(\mathrm{Viol}_{i,a,t_b,t}(\varphi) \wedge \neg \mathsf{F}^{+} t'.(\mathrm{RViol}_{i,a,t_b,t,t'}(\varphi))}$

Punished violations are defined analogously, with $P_{i,a}$ instead of $R_{i,a}$.

Using these violation modalities, we can define obligation and prohibition operators. The intuition between the prohibition operator is that something is forbidden if doing it leads to a violation to act. For obligation this intuition is reversed, namely that not doing φ before the deadline leads to a violation of omission. These definitions are adapted from [5,4].

Definition 3.3

$M, s \models_\tau F_{i,a,t_b}(\varphi) \iff$
$M, s \models_\tau (E_a\varphi \rightarrow \mathsf{AX}^{+}(t_v.\mathrm{Viol}^{a}_{i,a,t_b,t_v}(\varphi)))$
$M, s \models_\tau O_{i,a,t_b}(\varphi < \delta) \iff$ for all $\sigma \in \mathit{paths}(\mathcal{W}, \mathcal{R}, s),$ there exists
$j > \mathit{depth}(\mathcal{W}, \mathcal{R}, s), M, \sigma(j) \models_\tau \delta$ and for all $\mathit{depth}(\mathcal{W}, \mathcal{R}, s) \leq k < j$:
$M, \sigma(k) \models_\tau t_v.\neg\mathrm{Viol}^{o}_{i,a,t_b,t_v}(\varphi) \wedge \neg\delta$ and if for all $\mathit{depth}(\mathcal{W}, \mathcal{R}, s) \leq k < j$:
$M, \sigma_k \models_\tau \neg E_a\varphi$ then $M, \sigma_j \models_\tau t_v.\mathrm{Viol}^{o}_{i,a,t_b,t_v}(\varphi)$

4 Conclusion

A longer version can be found at `https://arxiv.org/abs/2205.10295`.

Thanks to the properties of these violations and obligations/prohibitions, we can also go from a general specification of a norm to a more specific instantiation of an obligation/prohibition, using a similar approach as [11,10] for the denotation of the norms. This specification would also allow us to use the violations of the agent itself as well as other agents to specify reparation, punishment, and meta norms.

One of the problems in our current system, is the way in which we mark a (lack of) action to cause a violation. This is due to two reasons. The first of this is that we want some notion of causality in our logic, but this is not very natural to express in temporal logics. A system where we could more directly talk about the actions an agent takes and their consequences could solve this.

The second reason is that we currently use implication over time to mark certain effects of actions as causing violations. However, this means we also bring in all the traditional problems of the material conditional. An alternative approach, such as using counts-as rules [8] could help in this case. However, most of those are based on notions of context, and how the context should operate in a branching temporal logic is not trivial and has not been investigated.

Other future work that can be done based on this system is looking into how we can specify more complex normative systems using this framework. Currently, we do not link norms explicitly to one another, but we expect that this can be done. In that case, a normative system could be specified as a graph, where all the nodes are norms. One of the questions that raises is what properties these graphs would need to get a fully enclosed normative system.

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Logics of true belief

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Abstract

In epistemic logic, the beliefs of an agent are modeled in a way very similar to knowledge, except that they are fallible. Thus, the pattern of an agent's *true* beliefs is an interesting subject for study. True beliefs have only been studied in an unsystematic and rather ambiguous way before, so in this paper, we conduct a systematic study on a novel modal logic with the boxdot operator $\boxdot$ as the only primitive modality, where $\boxdot\phi$, defined as the bundle $\Box\phi \wedge \phi$, explicitly captures the notion of true belief. With the help of a novel notion of $\boxdot$-bisimulation, we characterize the expressivity of this new language, and offer various completeness results; in particular, in accordance with previous works, we prove explicitly that $\mathbf{S4.4}^{\boxdot}$ is the "logic of true belief" for an agent with **KD45** beliefs. Moreover, we also discuss some interesting connections between our work and previous works on reflexive-insensitive logics and the so-called boxdot conjecture.

Keywords: Epistemic logic, true belief, completeness, bisimulation

1 Introduction

In epistemic logic, following the tradition initiated by Hintikka in [7], when we say that an agent *believes* that p, our expression can be interpreted as: p is true in every epistemic possible world which is possible for the agent, according to all the information (true or false) she possesses. Interpreted this way, beliefs seem very similar to knowledge, except that they are fallible.

Then, it seems interesting to study the behavior of an agent's *true* beliefs. Such a thesis has been touched upon before. For example, it is proved in [9] that $\mathbf{KD45}^{\mathcal{B}} \oplus \mathcal{K}\phi \leftrightarrow (\mathcal{B}\phi \wedge \phi)$ is deductively equivalent to $\mathbf{S4.4}^{\mathcal{K}} \oplus \mathcal{B}\phi \leftrightarrow \neg\mathcal{K}\neg\mathcal{K}\phi$, from which it is concluded that **S4.4** characterizes the true beliefs of an **KD45** agent; in [11], **S4.4** is also called "the logic of true belief" because it corresponds to the minimal extension of **KD45**-frames. [2] However, in these works, the notion of "true belief" is not singled out and studied independently, and the results concerning the behavior of true beliefs are presented in a rather unsystematic and ambiguous way. Hence, in this research, we will try to offer

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[2] I would like to thank an anonymous reviewer for introducing these results to me.

a systematic study of the logics of true belief through studying a new bundle operator $\boxdot\phi := \Box\phi \land \phi$ [3], which captures the notion of "true belief" explicitly.

2 Semantics and Expressivity

The new language $\mathcal{L}^{\boxdot}$ which takes $\boxdot$ has its only primitive modality and its semantics $\vDash_{\boxdot}$ are defined as follow:

Definition 2.1 ($\mathcal{L}^{\boxdot}$) $\mathcal{L}^{\boxdot}$-formulas are defined recursively as:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \land \phi \mid \boxdot\phi$$

Where $p \in \mathbf{P}$ ($\mathbf{P}$ is a set of atomic propositions).

$\diamonddot\phi$ is defined as $\neg \boxdot \neg\phi$, and $\lor$, $\to$, $\leftrightarrow$, $\bot$ are defined as usual.

Definition 2.2 ($\vDash_{\boxdot}$) For any relational model $\mathcal{M} = \langle W, \to, V\rangle$, for any $w \in W$, the semantics for $p \in \mathbf{P}$, $\top$ and the Boolean formulas are defined as usual; and the semantics for $\boxdot\phi$ is defined as follow:

$$\mathcal{M}, w \vDash_{\boxdot} \boxdot\phi \iff \mathcal{M}, w \vDash_{\boxdot} \phi, \text{ and for all } v \text{ s.t. } w \to v,\ \mathcal{M}, v \vDash_{\boxdot} \phi$$

It is easy to see that the expressivity of $\mathcal{L}^{\boxdot}$ is strictly weaker than $\mathcal{L}^{\Box}$, the standard modal language. But the expressivity of $\mathcal{L}^{\boxdot}$ can be characterized in a finer way with the help of the notion of $\boxdot$-bisimulation:

Definition 2.3 ($\boxdot$-bisimulation) For all models $\mathcal{M}$ and $\mathcal{N}$, a $\boxdot$-bisimulation between them is a nonempty binary relation $Z \subseteq W_{\mathcal{M}} \times W_{\mathcal{N}}$, such that for all $w \in W_{\mathcal{M}}, v \in W_{\mathcal{N}}$ such that wZv:

(Invariance) $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$;

($\boxdot$Zig) for all $w' \in W_{\mathcal{M}}$ such that $w \to w'$, $w'Zv$, or there is a $v' \in W_{\mathcal{N}}$ such that $v \to v'$ and $w'Zv'$;

($\boxdot$Zag) the definition is symmetric.

For two models $\mathcal{M}, w$ and $\mathcal{N}, v$, we say that $\mathcal{M}, w$ and $\mathcal{N}, v$ are $\boxdot$-bisimilar, denoted as $\mathcal{M}, w \leftrightarroweq_{\boxdot} \mathcal{N}, v$, iff there is a $\boxdot$-bisimulation Z between $\mathcal{M}$ and $\mathcal{N}$ such that wZv.

We show that $\boxdot$-bisimulation characterizes the expressivity of $\mathcal{L}^{\boxdot}$ through proving the following theorems:

Theorem 2.4 *For all models $\mathcal{M}, w$ and $\mathcal{N}, v$, $\mathcal{M}, w \leftrightarroweq_{\boxdot} \mathcal{N}, v \implies \mathcal{M}, w \equiv_{\mathcal{L}^{\boxdot}} \mathcal{N}, v$.*

Theorem 2.5 *For all $\boxdot$-saturated models $\mathcal{M}, w$ and $\mathcal{N}, v$, $\mathcal{M}, w \leftrightarroweq_{\boxdot} \mathcal{N}, v \iff \mathcal{M}, w \equiv_{\mathcal{L}^{\boxdot}} \mathcal{N}, v$.*

Where the notion of $\boxdot$-saturation is defined as follow:

Definition 2.6 ($\boxdot$-saturation) A model $\mathcal{M}$ is $\boxdot$-saturated, iff for all $w \in W_{\mathcal{M}}$, for all set Γ of $\mathcal{L}^{\boxdot}$-formulas, if Γ is finitely satisfiable in $\{w\} \cup \{v \mid w \to v\}$, then Γ is satisfiable in $\{w\} \cup \{v \mid w \to v\}$.

---

[3] The notation $\boxdot$ is not our invention. It is first introduced in [1] as an abbreviation.

Further, we prove a van Benthem-like characterization theorem for $\mathcal{L}^{\boxdot}$:

Theorem 2.7 *For all modal formula ϕ, ϕ is equivalent to a $\mathcal{L}^{\boxdot}$-formula $\iff$ ϕ is invariant under $\leftrightarroweq_{\boxdot}$, i.e. $\mathcal{L}^{\boxdot}$ is the fragment of $\mathcal{L}^{\Box}$ which is invariant under $\leftrightarroweq_{\boxdot}$.*

This shows that $\mathcal{L}^{\boxdot}$ is the fragment of $\mathcal{L}^{\Box}$ which is invariant under $\leftrightarroweq_{\boxdot}$.

3 Axiomatizations

The minimal system of $\mathcal{L}^{\boxdot}$, namely $\mathbf{T}^{\boxdot}$, is defined as follow:

Definition 3.1 ($\mathbf{T}^{\boxdot}$) $\mathbf{T}^{\boxdot}$ is the smallest set of $\mathcal{L}^{\boxdot}$-formulas which contains all the propositoinal tautologies and the axioms $\mathtt{K}^{\boxdot}$ and $\mathtt{T}^{\boxdot}$, and is closed under modus ponens, uniform substitution and $\mathtt{NEC}^{\boxdot}$, where $\mathtt{K}^{\boxdot}$, $\mathtt{T}^{\boxdot}$ and $\mathtt{NEC}^{\boxdot}$ are defined as follow:

$$\mathtt{K}^{\boxdot}\ \boxdot(p\to q)\to(\boxdot p\to\boxdot q)\qquad \mathtt{T}^{\boxdot}\ \boxdot p\to p\qquad \mathtt{NEC}^{\boxdot}\ \vdash\phi\implies\vdash\boxdot\phi$$

And the more general notion of $\boxdot$-logics is defined as:

Definition 3.2 A set $\mathbf{L}$ of $\mathcal{L}^{\boxdot}$-formulas is a $\boxdot$-logic, iff $\mathbf{L}$ is closed under modus ponens, uniform substitution and $\mathtt{NEC}^{\boxdot}$, and $\mathbf{T}^{\boxdot}\subseteq\mathbf{L}$.

Due to the affinity between $\boxdot$-logics and standard modal logics, the following lemma holds, from which a bunch of completeness results follows uniformly:

Lemma 3.3 *For any normal modal logic $\mathbf{L}$ such that $\mathbf{T}\subseteq\mathbf{L}$, and any reflexive class of models C, the following are true:*

(i) $\mathbf{L}^{\boxdot}=\{d(\phi)\mid\phi\in\mathbf{L}\}$ *is a $\boxdot$-logic;*

(ii) If $\mathbf{L}$ is sound / weakly complete / strongly complete w.r.t. C, then so is $\mathbf{L}^{\boxdot}$;

Theorem 3.4

- $\mathbf{T}^{\boxdot}$ *is sound and strongly complete w.r.t. the class of reflexive frames;*
- $\mathbf{S4}^{\boxdot}$ *is sound and strongly complete w.r.t. the class of reflexive and transitive frames;*
- $\mathbf{KTB}^{\boxdot}$ *is sound and strongly complete w.r.t. the class of reflexive and symmetric frames;*
- $\mathbf{S5}^{\boxdot}$ *is sound and strongly complete w.r.t. the class of reflexive, transitive and symmetric frames.*

And based on the expressivity results we have offered, the above theorem can be further generalized to:

Theorem 3.5

- $\mathbf{T}^{\boxdot}$ *is sound and strongly complete w.r.t. the class of all frames, the class of serial frames, and the class of irreflexive frames;*
- $\mathbf{S4}^{\boxdot}$ *is sound and strongly complete w.r.t. the class of transitive frames, the class of serial and transitive frames, and the class of irreflexive and transitive frames;*

- **KTB**$^{\boxdot}$ *is sound and strongly complete w.r.t. the class of symmetric frames, the class of serial and symmetric frames, and the class of irreflexive and symmetric frames.*
- **S5**$^{\boxdot}$ *is sound and strongly complete w.r.t. the class of symmetric and transitive frames, the class of serial, symmetric and transitive frames, and the class of the class of irreflexive, symmetric and transitive frames*

In particular, we also show that the following theorem concerning provability logic follows from our results:

Theorem 3.6 [4] **Grz**$^{\boxdot}$ *is sound and strongly complete w.r.t. the class of* **GL**-*frames, where* **Grz** *is the system obtained by adding* $\mathtt{Grz}: \Box(\Box(p \to \Box p) \to p) \to p$ *to* **K**, *and a* **GL**-*model is one that is transitive and conversely well-founded.*

We also offer the axiomatizatoin of the $\boxdot$-logic over the class of **KD45**-frames (which coincides with the results in [9] and [11]):

Theorem 3.7 **S4.4**$^{\boxdot}$ *is sound and strongly complete w.r.t. the class of* **KD45**-*frames.*

The theorem follows from the following two propositions and the fact that transitivity and weak euclidicity are preserved under reflexive closure.

Proposition 3.8 *Every* **KD45**-*model is* $\boxdot$-*bisimilar to a model which is transitive and weakly euclidean (that is, for all* $w, v, u \in W_{\mathcal{M}}$, wRv, wRu *and* $w \neq u$ *implies that* vRu*), and vice versa.*

Proposition 3.9 **K4.4** *is sound and strongly complete w.r.t. the class of transitive and weakly euclidean frames.*

From the above proof, a general strategy to axiomatize $\boxdot$-logics over classes of non-reflexive frames is also concluded, through which we can also prove that the $\boxdot$-logic that corresponds to **K5**-frames is $(\mathbf{T} \oplus .4 \oplus p \to \Box(\Diamond \Box p \to p))^{\boxdot}$.

Further, we show that if a **KD45** agent has **S5**$^{\boxdot}$ true beliefs, then the notions of belief and knowledge coincide for her, through proving the following proposition:

Proposition 3.10 *For any subclass* C *of the class of* **KD45**-*frames,* **S5**$^{\boxdot}$ *is sound w.r.t.* C *iff* C *includes only* **S5**-*frames.*

Which may shed light on relevant philosophical discussions.

4 Connections with previous works

We show that the logics of true belief we developed above are closely related to the so-called "reflexive-insensitive logics" (*RI*-logics) [5], which takes $\circ\phi := \phi \to \Box\phi$ as its primitive modality.

[4] In [1], Boolos has also proved that $\mathbf{GL} \vdash t_{\Box}(d(\phi))$ iff $\mathbf{Grz} \vdash \phi$ (expressed in our notation), which is very similar to our result here.

[5] Which is studied in [10], [13], [12], [5], [4], [6] and [2].

Based on the fact that $\boxdot$ and $\circ$ are inter-definable, every results concerning the expressivity of $\mathcal{L}^{\boxdot}$ also holds for $\mathcal{L}^{\circ}$. And we also show that the axiomatizations of *RI*-logics achieved via the technique of "$\circ$-translation" (defined in [5]) is closely related to the $\boxdot$-logics we discussed above.

We also offer a new conceptual interpretation of the notion of boxdot property (BDP) defined in [8], which generalizes the so-called "boxdot conjecture" [6] :

Definition 4.1 (boxdot property (BDP)) A normal modal logic $\mathbf{L_0}$ has BDP, iff For all normal modal logic $\mathbf{L}$, if $\{\phi \in \mathcal{L}^{\Box} \mid t(\phi) \in \mathbf{L}\} = \mathbf{L_0}$, then $\mathbf{L} \subseteq \mathbf{L_0}$. Where $t(\phi)$ interprets every $\Box$ in ϕ with a $\boxdot$ (interpreted as an abbreviation).

We show that what it means conceptually for a normal modal logic $\mathbf{L_0}$ extending $\mathbf{T}$ to have BDP is that, if the $\boxdot$-bisimulation-invariant fragment of a normal modal logic $\mathbf{L}$ is equivalent to that fragment of $\mathbf{L_0}$, then $\mathbf{L}$ itself must be a subset of $\mathbf{L_0}$: in other words, if $\mathbf{L}$ can say something more about the models than $\mathbf{L}$, then that "something more" must involve some property which can be preserved under $\boxdot$-bisimulation.

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[6] Which is studied in [3], [14], [15], [8] and [6].